SOLUTION SET

Chapter 4

RADIATIVE TRANSITIONS AND EMISSION LINEWIDTH

"LASER FUNDAMENTALS"

Second Edition

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1. Show how to solve the differential equation (4.1) for N_u to obtain (4.2). Show that your answer is in fact a solution of the equation.

$$\frac{dN_u}{dt} = -A_{ue} N_u$$

Re-write: $$\frac{dN_u}{N_u} = -A_{ue}\, dt$$

$$\int \frac{dN_u}{N_u} = -A_{ue} \int dt$$

$$\ln N_u = -A_{ue} t \Rightarrow e^{\ln N_u} = e^{-A_{ue} t}$$

or $$\underline{N_u = N_u^0 e^{-A_{ue} t}}$$

substitute into top equation:

$$\frac{dN_u}{dt} = -A_{ue} N_u^0 e^{-A_{ue} t} = -A_{ue} N_u$$

$$\therefore \; -A_{ue} N_u = -A_{ue} N_u \qquad \text{Q.E.D.}$$

2. Compute the power radiated from an electron oscillating at an angular frequency of $\omega = 10^{15}$ rad/s. Assume that x_0 is equal to the Bohr radius of an orbiting electron.

From (4.11)

$$P_R = -\frac{dE_T}{dT} = \frac{16\pi^3 \nu_0^4 (e x_0)^2}{3\epsilon_0 c^3} = \frac{\omega_0^4 (e x_0)^2}{3\pi \epsilon_0 c^3}$$

for $\omega_0 = 10^{15}$ rad/sec $\quad x_0 = a_H = 5.3 \times 10^{-11}$ m

$$P_R = \frac{(10^{15})^4 (1.6\times10^{-19} C \; 5.3\times10^{-11} m)^2}{3\pi \; (8.85\times10^{-12} F/m)\,(3\times10^8 m/s)^3}$$

$$= \underline{3.19\times10^{-14} \; J/s = 3.19\times10^{-14} \text{ Watts}}$$

3. If 10 Torr of He gas is added to an electrical discharge composed of H atoms, how much would the decay time of the $n = 2$ state of H decrease if the pressure-broadening factor due to He is 10^7/s-Torr? What would it be if the He pressure were increased to 100 Torr? (Recall that 1 Torr = 1/760 of atmospheric pressure.) *Hint:* Use the average value of the transition probability for $n = 2$ to $n = 1$ of H calculated in the example at the end of this chapter.

From (4.101) $\quad A_{21} = 4.7 \times 10^8/s$

$$\tau_2 = \frac{1}{A_{21}} = \frac{1}{4.7 \times 10^8/s} = 2.13 \times 10^{-9}\,s$$

For pressure broadening

$$\gamma_{21} = A_{21} + \gamma_{Helium}$$

(a)
$$= 4.7 \times 10^8/s + (10\ Torr)(10^7/s \cdot Torr)$$

$$= 4.7 \times 10^8/s + 10^8/s = 5.7 \times 10^8/s$$

$$\tau_2 = \frac{1}{5.7 \times 10^8/s} = 1.75 \times 10^{-9}\,s$$

A decrease of $(2.13 - 1.75) \times 10^{-9}\,s = \underline{3.8 \times 10^{-10}\,s}$

(b)
$$\gamma_{21} = 4.7 \times 10^8/s + (100\ Torr)(10^7\,s/Torr)$$

$$= 4.7 \times 10^8/s + 10^9/s = 1.47 \times 10^9/s$$

$$\tau_2 = \frac{1}{1.47 \times 10^9/s} = 6.8 \times 10^{-10}\,s$$

A decrease of $(2.13 - 0.68) \times 10^{-9}\,s = \underline{1.45 \times 10^{-9}\,s}$

4. With reference to (4.24), obtain values of the frequency at which the intensity drops to one half its maximum value. What is the FWHM (full width at half maximum) value for the frequency width?

From (4.24)

$$I(\omega) = I_0 \frac{\gamma_0/2\pi}{(\omega-\omega_0)^2 + \gamma_0^2/4}$$

$$I(\omega) = I_{max} \text{ at } \omega = \omega_0$$

$$\therefore I_{max} = \frac{2}{\pi \gamma_0} I_0$$

Solve for ω where $I(\omega) = \frac{1}{2} I_{max} = \frac{1}{2}\left(\frac{2}{\pi\gamma_0}\right) I_0 = \frac{I_0}{\pi\gamma_0}$

$$I(\omega) = I_0 \frac{\gamma_0/2\pi}{(\omega-\omega_0)^2 + \frac{\gamma_0^2}{4}} = \frac{I_0}{\pi\gamma_0}$$

$$\frac{\gamma_0}{2\pi\left[(\omega-\omega_0)^2 + \gamma_0^2/4\right]} = \frac{1}{\pi\gamma_0}$$

$$\frac{\gamma_0^2}{2} = (\omega-\omega_0)^2 + \gamma_0^2/4$$

$$(\omega-\omega_0)^2 = \frac{\gamma_0^2}{4} \Rightarrow \omega-\omega_0 = \pm\frac{\gamma_0}{2} \Rightarrow \omega = \omega_0 \pm \frac{\gamma_0}{2}$$

$$\Delta\omega_{FWHM} = 2\,|(\omega-\omega_0)| = 2\,\frac{\gamma_0}{2} = \gamma_0$$

$$\therefore \underline{\Delta\omega_{FWHM} = \gamma_0}$$

5. Using the value of $I(\omega)$ in (4.24), show that $\int_0^\infty I(\omega)\,d\omega = I_0$ (eqn. 4.25). *Hint:* You need to assume that $\gamma_{ul} \ll \omega_0$, a reasonable approximation.

$$I(\omega) = I_0 \frac{\gamma_0/2\pi}{(\omega-\omega_0)^2 + \gamma_0^2/4}$$

$$\int_0^\infty I(\omega)\,d\omega = \frac{I_0\gamma_0}{2\pi}\int \frac{d\omega}{(\omega-\omega_0)^2+\gamma_0^2/4}$$

let $x = \frac{2(\omega-\omega_0)}{\gamma_0}$

$dx = \frac{2}{\gamma_0}d\omega$

Then $$\int_0^\infty I(\omega)\,d\omega = \frac{I_0}{\pi}\int_{-2\frac{\omega_0}{\gamma_0}}^{\infty} \frac{dx}{x^2+1}$$

<u>First method</u>

but for $\omega \gg \gamma_0$ $\quad -\frac{2\omega_0}{\gamma_0} \to -\infty$

Hence $$\int_0^\infty I(\omega)\,d\omega \cong \frac{I_0}{\pi}\int_{-\infty}^{\infty}\frac{dx}{x^2+1} \quad \text{but} \int_{-\infty}^{\infty}\frac{dx}{x^2+1} = \pi$$

Therefore $$\int_0^\infty I(\omega)\,d\omega = \frac{I_0\pi}{\pi} = \underline{I_0}$$

<u>Second method</u>

$$\int_{-\frac{2\omega_0}{\gamma_0}}^{\infty}\frac{dx}{x^2+1} = \int_{-2\omega_0/\gamma_0}^{0}\frac{dx}{x^2+1} + \int_0^\infty \frac{dx}{x^2+1} = \tan^{-1}x\Big|_{x=-\frac{2\omega_0}{\gamma_0}}^{x=0} + \frac{\pi}{2}$$

$$= 0 + \tan^{-1}\left(-\frac{2\omega_0}{\gamma_0}\right) + \frac{\pi}{2} = \pi \text{ for } \omega_0 \gg \gamma_0$$

Therefore $$\int_0^\infty I(\omega)\,d\omega = \frac{I_0\pi}{\pi} = \underline{I_0}$$

6. Compute the Doppler broadening for the 632.8-nm laser transition in the He–Ne laser, assuming a single isotope of Ne^{20} and that the laser operates at a discharge-bore temperature of 100°C. Compare this broadening to the natural broadening obtained in the example at the beginning of the chapter.

$\lambda = 632.8$ nm $\quad T = 373$ K $\quad M_N = 20$

$$\Delta\nu_D = 7.16 \times 10^{-7}\, \nu_0 \sqrt{\frac{T}{M_N}} \qquad (4.59)$$

$$= 7.16 \times 10^{-7}\, \frac{3 \times 10^8 \text{ m/s}}{632.8 \times 10^{-9} \text{ m}} \sqrt{\frac{373}{20}}$$

$$= \underline{1.47 \times 10^9 \text{ Hz}}$$

From Table 4-1 (pg 114) it can be seen that $\Delta\nu_N = 1.4 \times 10^7$ Hz for that transition.

Hence $\dfrac{\Delta\nu_D}{\Delta\nu_N} = \dfrac{1.47 \times 10^9 \text{ Hz}}{1.4 \times 10^7 \text{ Hz}} = \underline{105}$

Therefore Doppler broadening increases the linewidth by more than <u>two orders of magnitude</u>.

7. Assume that you have the following two special isotope mixtures of cadmium:
   (a) equal amounts of Cd^{112} and Cd^{114};
   (b) equal amounts of Cd^{110} and Cd^{114}.

   Assume that the He–Cd laser is operating in a gaseous discharge at a temperature of 300°C. Make a plot of the emission envelope of the 441.6-nm transition for isotope mixtures (a) and (b). Use the following values for the difference in the center frequencies of the emission of the different isotopes on the 441.6-nm transition:

$$\Delta\nu(Cd^{110} - Cd^{112}) = 1.57 \times 10^9 \text{ Hz};$$

$$\Delta\nu(Cd^{112} - Cd^{114}) = 1.46 \times 10^9 \text{ Hz}.$$

Note: Both isotopes radiate at a wavelength of approximately 441.6 nm, with only a very slight difference in their center frequencies as indicated here.

The Doppler width of Cd atoms and ions at 300 °C or 573 K (from 4.59) is

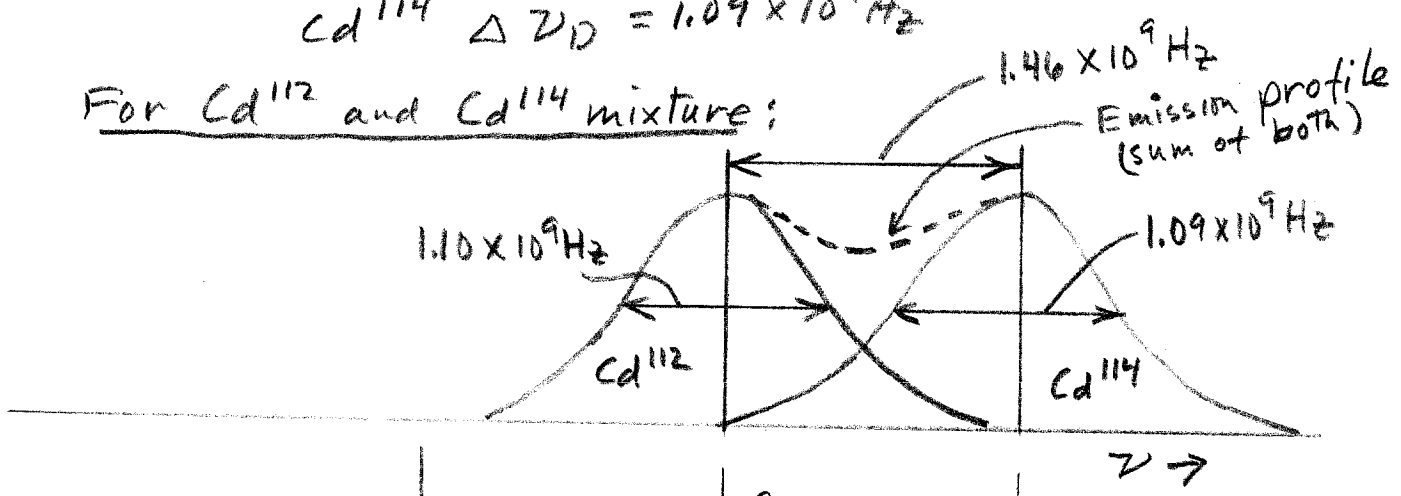
$$\Delta\nu_D = (7.16\times10^{-7})\frac{3\times10^8 \text{ m/s}}{441.6\times10^{-9}\text{m}}\sqrt{\frac{573}{M_N}} = \frac{1.16\times10^{10}}{\sqrt{M_N}}$$

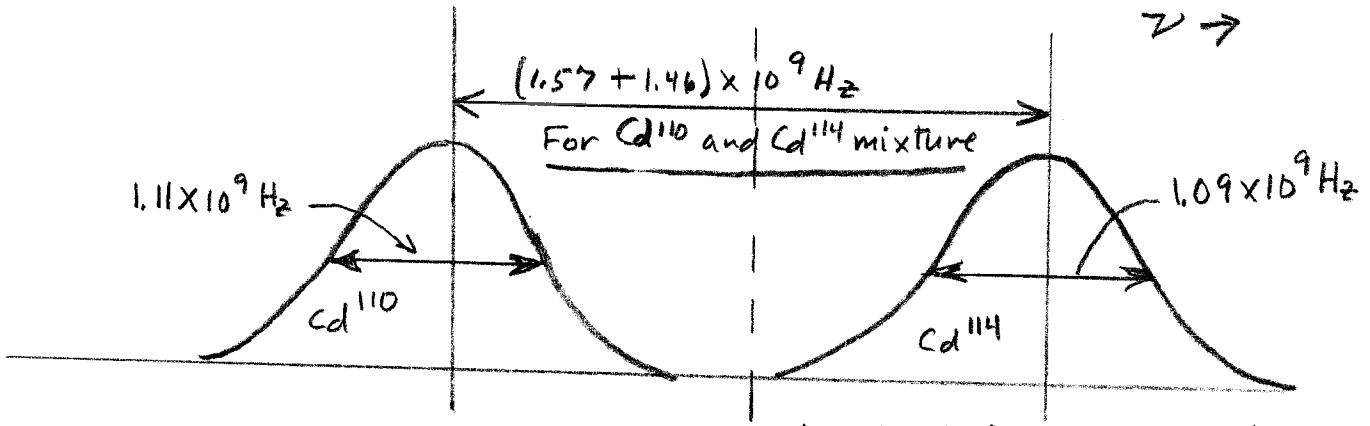
For: Cd^{110} $\Delta\nu_D = 1.11\times10^9$ Hz

Cd^{112} $\Delta\nu_D = 1.10\times10^9$ Hz

Cd^{114} $\Delta\nu_D = 1.09\times10^9$ Hz

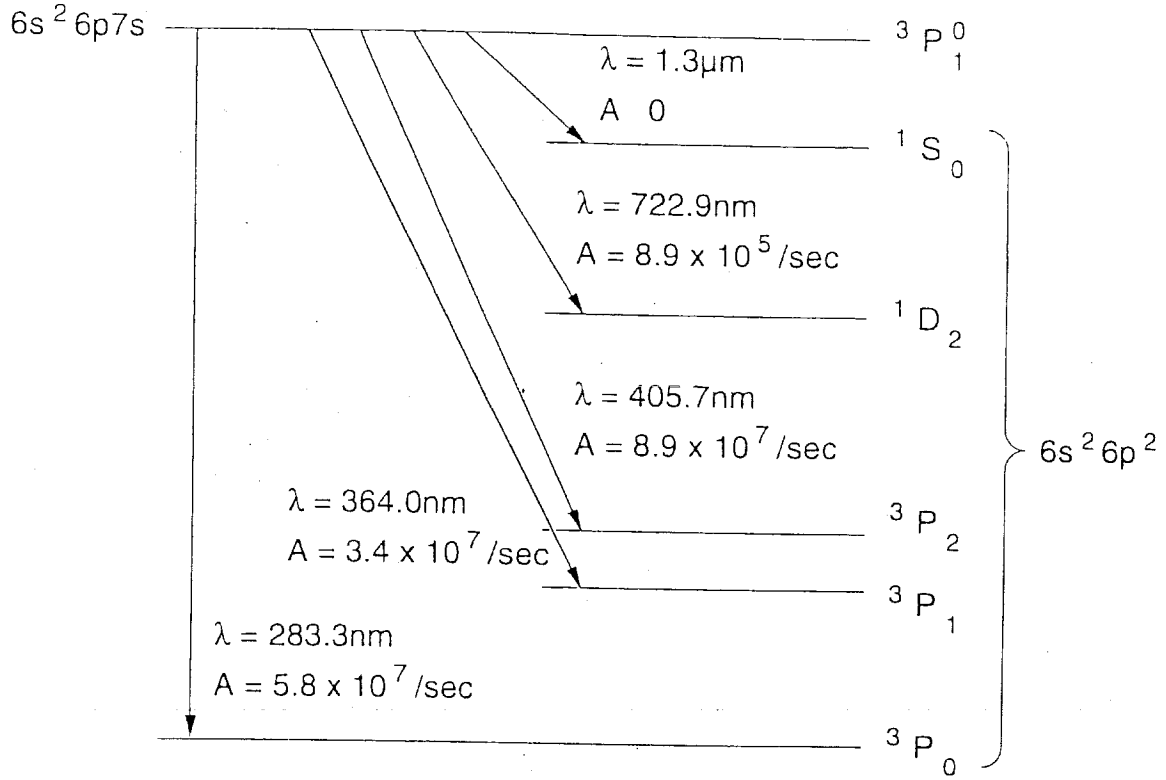
For Cd^{112} and Cd^{114} mixture:





Emission profile is just individual isotope profiles

8. The pulsed lead laser operates on several transitions in the visible and ultraviolet spectrum, including the 722.9-nm and 405.7-nm transitions. Compute the natural linewidth, the Doppler width, and $\Delta\nu_D/\nu_0$ for these two laser transitions. Assume that the lead vapor is at a temperature of 1,300 K, providing a lead vapor pressure of approximately 1 Torr. The energy-level diagram associated with those transitions is shown in the figure. Why would the $6p7s\ ^3P_1^o \rightarrow 6p^2\ ^1D_2$ transition not



The dominant isotope for lead is Pb^{208}.

normally be allowed according to electric dipole transition selection rules? (This is a case where the rules break down; there is a finite value of the electric dipole matrix element connecting these two energy levels, because the $^1D_2$ level is actually not a pure singlet state but instead has some triplet wave function mixed in with the singlet wave function. This often occurs for heavier atoms.)

For 722.9 nm $\quad \Delta\nu_D = 7.16 \times 10^{-7} \frac{3\times10^8}{722.9\times10^{-9}} \sqrt{\frac{1300}{208}} = \underline{7.42\times10^8\ Hz}$

For 405.7 nm $\quad \Delta\nu_D = 7.16\times10^{-7} \frac{3\times10^8}{405.7\times10^{-9}} \sqrt{\frac{1300}{208}} = \underline{1.32\times10^9\ Hz}$

Both transitions have no radiative decay from their lower level

∴ Natural linewidths are the same

$$\Delta\nu_N = \frac{\sum A_{ni}}{2\pi} = (5.8 + 3.4 + 8.9 + .0089)\times10^7/2\pi$$

$$= \underline{2.88\times10^7\ Hz}$$

<u>Doppler dominates</u>

9. The selenium soft–X-ray laser operates at a wavelength of 20.6 nm in Se^{24+} ions. Those ions are produced at a plasma temperature of approximately 10^7 K. How large is the Doppler width of that laser transition compared with the width of the He–Ne 632.8-nm laser determined in Problem 6?

From (4.59)

$$\Delta\nu_D = 7.16 \times 10^{-7}\, \nu_0 \sqrt{\frac{T}{M_N}}$$

From The Appendix, The mass number for Se is 80

$$T = 10^7 K \qquad \nu_0 = \frac{c}{\lambda} = \frac{3 \times 10^8 m/s}{20.6 \times 10^{-9} m} = 1.46 \times 10^{16} Hz$$

$$\therefore \quad \Delta\nu_D = 7.16 \times 10^{-7}\, 1.46 \times 10^{16} \sqrt{\frac{10^7}{80}}$$

$$= \underline{3.70 \times 10^{12} Hz}$$

He-Ne 632.8 nm laser has a Doppler width (from Table 4-1) of

$$\underline{1.5 \times 10^9 Hz}$$

Therefore Se soft x-ray laser has a Doppler width of greater Than 3 orders of magnitude greater Than He-Ne laser.

10. If two emission lines in two different species have the same center frequency ν_0 and the same emission linewidth $\Delta\nu_{\text{FWHM}}$, but one is homogeneously broadened and the other is Doppler broadened, how much greater is the value of the emission from the homogeneously broadened line at a frequency displaced from ν_0 by a factor of $\nu = 5\Delta\nu_{\text{FWHM}}$?

Assume both emission lines have the same total emission intensity I_0 over the entire emission spectrum of each transition. Let $\Delta\nu_{FWHM}$ be the same for both transitions and let $\Delta\nu_{FWHM} = \Delta\nu_F$ for simplicity. Then evaluate both emissions at $(\nu-\nu_0) = 5\Delta\nu_F$

For the homogeneous transition, from (4.57) and $\gamma_{ul}^T = 2\pi\Delta\nu_F$ we have

$$I^H(5\Delta\nu_F) = I_0 \frac{\frac{2\pi\Delta\nu_F}{4\pi^2}}{(5\Delta\nu_F)^2 + \left(\frac{2\pi\Delta\nu_F}{4\pi}\right)^2} = \frac{\frac{\Delta\nu_F}{2\pi}}{\Delta\nu_F^2\left[5^2 + \left(\frac{1}{2}\right)^2\right]}$$

$$= \frac{I_0}{\Delta\nu_F\, 2\pi[25.25]} = \underline{\frac{6.30\times10^{-3}}{\Delta\nu_F} I_0}$$

For the Doppler transition, from (4.60) and $\Delta\nu_D = \Delta\nu_F$

$$I^D(5\Delta\nu_F) = \frac{2}{\Delta\nu_F}\left(\frac{\ln 2}{\pi}\right)^{1/2} I_0\, e^{-\frac{4\ln 2(5\Delta\nu_F)^2}{\Delta\nu_F^2}}$$

$$= \frac{1}{\Delta\nu_F}\left(\frac{4\ln 2}{\pi}\right)^{1/2} I_0\, e^{-100\ln 2}$$

$$= \frac{7.41\times10^{-31}}{\Delta\nu_F} I_0$$

$$\therefore \text{Ratio} = \frac{I^H}{I^D} = \frac{6.30\times10^{-3}}{7.41\times10^{-31}}\left(\frac{\Delta\nu_F}{\Delta\nu_F}\right)\frac{I_0}{I_0} = \underline{8.50\times10^{27}}$$

11. Obtain the expression for the oscillator strength of (4.7) from (4.78).

From (4.78)

$$A_{ul} = \frac{2\pi e^2}{\epsilon_0 m_e c \lambda_{ul}^2} \frac{g_l}{g_u} f_{lu}$$

rewriting:

$$A_{ul} = \frac{2\pi e^2 f_{lu}}{\epsilon_0 m_e c \left(\frac{g_u}{g_l}\right) \lambda_{ul}^2}$$

but

$$\frac{2\pi e^2}{\epsilon_0 m_e c} = \frac{2\pi (1.6\times10^{-19})^2}{8.85\times10^{-12}\, 9.1\times10^{-31}\, 3\times10^{8}}$$

$$= 6.66\times10^{-5}$$

$$= \frac{10^{-4}}{1.5}$$

hence

$$A_{ul} = \boxed{\frac{10^{-4} f_{lu}}{1.5 \left(g_u/g_l\right) \lambda_{ul}^2}}$$

12. The first nine levels of atomic calcium are tabulated as follows.

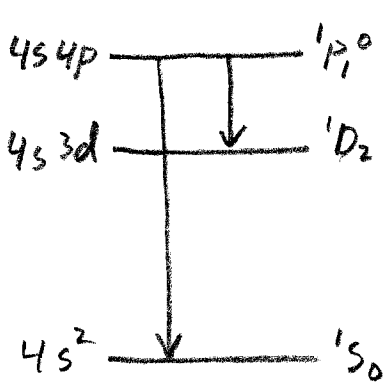
4s4p	$^1P_1^o$	2.3652×10^6 m^{-1}
	$^1D_2$	2.1850×10^6 m^{-1}
4s3d	$^3D_3$	2.0370×10^6 m^{-1}
	$^3D_2$	2.0349×10^6 m^{-1}
	$^3D_1$	2.0335×10^6 m^{-1}
	$^3P_2^o$	1.5316×10^6 m^{-1}
4s4p	$^3P_1^o$	1.5210×10^6 m^{-1}
	$^3P_0^o$	1.5158×10^6 m^{-1}
$4s^2$	$^1S_0$	0.00

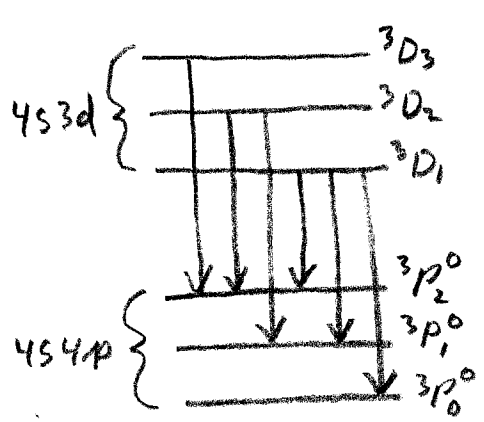
Determine which transitions are most likely to occur based upon the selection rules given in this chapter. Also determine the wavelengths of those transitions. The energy value for a specific level i above the ground level 1 is given on the right-hand side in wavenumbers σ_{i1}. The wavenumber σ_{ij} of the transition from level i to level j is determined from the equation for the energy difference between the two levels involved in the transition: $\Delta E_{ij} = E_i - E_j = h\nu_{ij} = hc\sigma_{ij} = hc(\sigma_{i1} - \sigma_{j1})$. Therefore, the wavelength of the transition would be $\lambda_{ij} = 1/\sigma_{ij} = 1/(\sigma_{i1} - \sigma_{j1})$.

selection rules: $\Delta l = \pm 1$ $\Delta S = 0$ $\Delta L = 0, \pm 1$ $\Delta J = 0, \pm 1$

allowed Transitions

$^1P_1^o \rightarrow {}^1D_2$	5.55 μm	$^1P_1^o \rightarrow {}^1S_0$	422.8 nm
$^3D_3 \rightarrow {}^3P_2^o$	1.979 μm	$^3D_2 \rightarrow {}^3P_2^o$	1.987 μm
$^3D_2 \rightarrow {}^3P_1^o$	1.946 μm	$^3D_1 \rightarrow {}^3P_2^o$	1.992 μm
$^3D_1 \rightarrow {}^3P_1^o$	1.951 μm	$^3D_1 \rightarrow {}^3P_0^o$	1.932 μm





CH 4

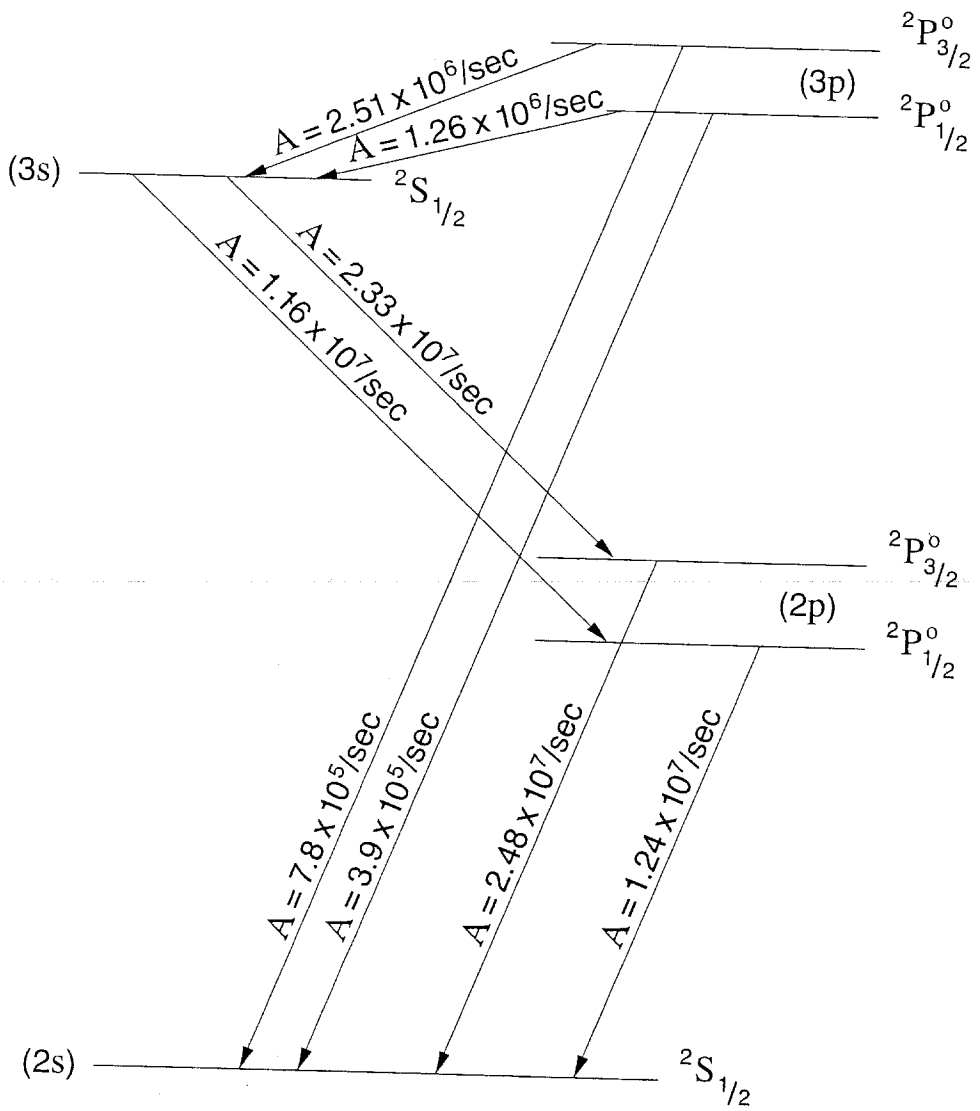
13. Shown in the first figure on page 134 are the first six energy levels of neutral lithium (Li). Determine the natural linewidth for the following transitions:

$$3p\,^2P^o_{3/2} \rightarrow 2s\,^2S_{1/2};$$

$$3s\,^2S_{1/2} \rightarrow 2p\,^2P^o_{1/2}.$$

Explain why there is no transition shown for

$$3s\,^2S_{1/2} \rightarrow 2s\,^2S_{1/2}.$$



13 (CONT.)

Li $1s^2 2s$ ground state

$$\Delta\nu_{ne}^{N} = \frac{1}{2\pi}\left[\sum_i A_{ni} + \sum_j A_{nj}\right]$$

for $3p\ ^2P^o_{3/2} \rightarrow 2s\ ^2S_{1/2}$

lower level is ground state

$\therefore \sum A_{nj} = 0$

$$\Delta\nu_N = \frac{1}{2\pi}\left[2.51\times10^6/s + 7.8\times10^5/s\right] = \underline{5.24\times10^5\ Hz}$$

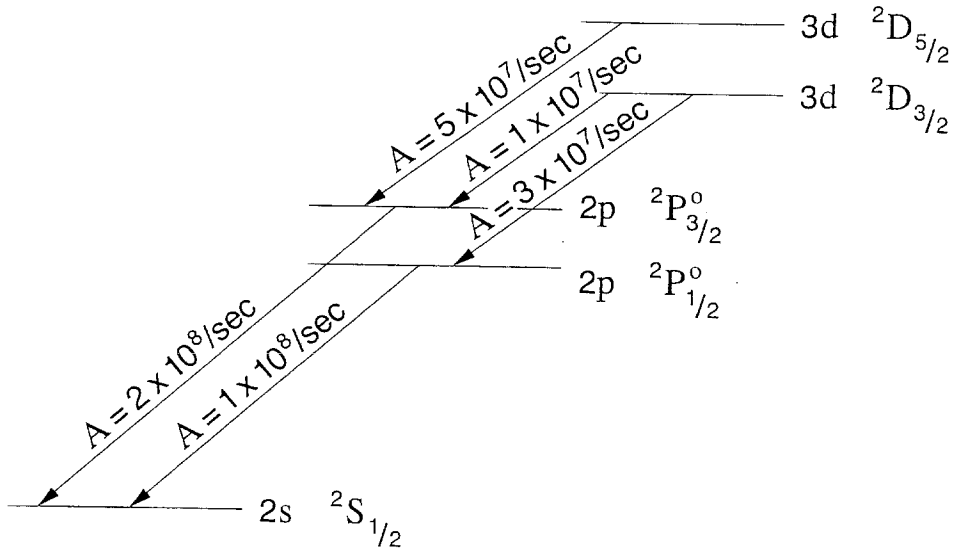
for $3s\ ^2S_{1/2} \rightarrow 2p\ ^2P^o_{1/2}$

$$\Delta\nu_N = \frac{1}{2\pi}\left[1.16\times10^7 + 2.33\times10^7 + 1.24\times10^7\right]$$

$$= \frac{4.73\times10^7}{2\pi} = \underline{7.53\times10^6\ Hz}$$

CH 4

14. Consider the set of energy levels and related transitions shown in the second figure on page 134. What is the natural emission linewidth of the $^2D_{3/2} \rightarrow {}^2P^o_{3/2}$ transition? If the species associated with these transitions is active within an electrically excited gaseous medium and if the emission shown in the diagram is produced at a gas pressure of 4 Torr, what would the pressure-broadening coefficient have to be (in MHz/Torr) in order for it to equal the natural emission linewidth?



$$\Delta\nu_N^u = \frac{\sum_i A_{ui}}{2\pi} = \frac{1\times10^7 + 3\times10^7}{2\pi} = \frac{4\times10^7}{2\pi} = 6.37\times10^6\,Hz$$

$$\Delta\nu_N^l = \frac{\sum_j A_{lj}}{2\pi} = \frac{2\times10^8}{2\pi} = 3.18\times10^7\,Hz$$

$$\Delta\nu_N = \Delta\nu_N^u + \Delta\nu_N^l = 6.37\times10^6 + 3.18\times10^7$$

$$= \underline{3.82\times10^7\,Hz}$$

$$B\,(MHz/Torr)\times 4\,Torr = 3.82\times10^7\,Hz = 38.2\,MHz$$

$$B = 9.55\times10^6/Torr = \underline{9.55\,MHz/Torr}$$

15. A new solid-state laser crystal is instantaneously pumped to its upper laser level. Emission is subsequently observed to occur over a FWHM emission linewidth of 100 nm at a peak emission wavelength of 700 nm. The emission is observed to decay with a measured lifetime of 200 μs up to a temperature of 200 K. As the temperature is increased above 200 K, the decay time decreases to a value of 150 μs at 300 K. The lower laser level decays only via collisions at a rate of 10^{13}/s. Determine the portion of the linewidth associated with each of the possible broadening mechanisms, and indicate which mechanism is the dominant mechanism.

$$\Delta\lambda_{FWHM} = 100\ \text{nm} \qquad \lambda_0 = 700\ \text{nm}$$

$$\Delta\nu_H = \frac{1}{2\pi}\left[\left(\sum_i A_{ui} + \sum_j A_{lj}\right) + \frac{1}{T_1^u} + \frac{1}{T_1^l} + \frac{2}{T_2}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{200\times10^{-6}} + \frac{1}{T_1^u} + 10^{12} + \frac{2}{T_2}\right]$$

$$\gamma_u = \sum_i A_{ui} + \frac{1}{T_1^u}$$

$$\frac{1}{150\times10^{-6}} = \frac{1}{200\times10^{-6}} + \frac{1}{T_1^u}$$

$$\therefore \frac{1}{T_1^u} = 1.667\times10^3 \text{ and } T_1^u = 6\times10^{-4}\ \text{s}$$

$$\therefore \frac{2}{T_2} = \left[2\pi\Delta\nu_H - \frac{1}{200\times10^{-6}} - 10^{13} - 1.667\times10^3\right]$$

$$= 2\pi\, 6.12\times10^{13} - 5\times10^3 - 10^{13} - 1.667\times10^3$$

$$\underline{\frac{2}{T_2} = 3.75\times10^{14}} \quad \underline{\text{dominant}}$$

$$\text{and } \Delta\nu = \frac{3.75\times10^{14}}{2\pi} = \underline{6.0\times10^{13}\ \text{Hz}}$$