SOLUTION SET

Chapter 6

RADIATION AND THERMAL EQUILIBRIUM – ABSORPTION AND STIMULATED EMISSION

"LASER FUNDAMENTALS"

Second Edition

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1. Calculate the number of radiation modes in a cube 1 mm on a side for a spread of 0.001 nm centered at 514.5 nm and a spread of 0.01 μm centered at 10.6 μm.

$$\rho(\nu) = \frac{8\pi\nu^3}{3c^3} \qquad \frac{d\rho(\nu)}{d\nu} = \frac{8\pi\nu^2}{c^3}$$

The number of modes within a volume V and frequency width $\Delta\nu$ is:

$$\frac{d\rho(\nu)}{d\nu} \cdot \Delta\nu \cdot V$$

(a) $V = 10^{-9}\ m^3$ $\quad \lambda = 514.5\ nm \Rightarrow \nu = 5.83\times10^{14}/s$

$\Delta\lambda = 0.001\ nm \Rightarrow \Delta\nu = \frac{c}{\lambda^2}\Delta\lambda = 1.13\times10^{9}$

$$\frac{8\pi\nu^2}{c^3} \cdot \Delta\nu \cdot V = \underline{3.58\times10^{5}\ \text{modes}}$$

(b) $V = 10^{-9}\ m^3$ $\quad \lambda = 10.6\ \mu m \Rightarrow \nu = 2.83\times10^{13}/s$

$\Delta\lambda = 0.01\ \mu m \Rightarrow \Delta\nu = 2.67\times10^{10}/s$

$$\frac{8\pi\nu^2}{c^3} \cdot \Delta\nu \cdot V = \underline{1.99\times10^{4}\ \text{modes}}$$

2. Consider a 1-mm–diameter surface area of carbon (graphite). Calculate how many atoms would exist in energy levels from which they could emit radiation at wavelengths shorter than 700 nm (visible light and shorter wavelengths) for surface temperatures of 300 K, 1,000 K, and 5,000 K when the solid is in thermal equilibrium at those temperatures. Assume that only those atoms within a depth of 10 nm of the material surface can emit observable radiation.

Density of carbon is 2.25 gm/cm^3

$$\therefore N = \frac{6 \times 10^{23} \text{ atoms/mole}}{12 \text{ gm/mole}} \frac{2.25 \text{ gm}}{\text{cm}^3} \frac{10^6 \text{ cm}^3}{\text{m}^3} = 1.13 \times 10^{29} \frac{\text{atoms}}{\text{m}^3}$$

$$N_T = \text{Total \# of atoms} = N \cdot V = 1.13 \times 10^{29} \frac{}{\text{m}^3} \pi (0.0005)^2 \text{m}^2 \, 10^{-8} \text{m}$$

$$= 8.87 \times 10^{14} \text{ carbon atoms in volume}$$

Visible light begins at 700 nm

$$E = h\nu = \frac{hc}{\lambda} = \frac{4.13 \times 10^{-15} \text{eV.s} \; 3 \times 10^8 \frac{\text{m}}{\text{s}}}{700 \times 10^{-9} \text{m}} = 1.77 \text{eV}$$

N_R = Total # radiating in visible and shorter λ

$$= \int_{1.77}^{\infty} N(E)\,dE = \int_{1.77}^{\infty} \frac{N}{kT} e^{-E/kT}\,dE$$

$$= \frac{N_T}{kT}(-kT)\Big|_{1.77\text{eV}}^{\infty}$$

$$= -N_T\left[0 - e^{-1.77/kT}\right] = \underline{e^{-\frac{1.77}{kT}} N_T}$$

(continued)

2 (cont.)

For $T = 300K$ $\quad kT = 8.63\times10^{-5}\frac{eV}{K}\,300K = 2.59\times10^{-2}\,eV$

$$N_R = 8.87\times10^{14}\,e^{-\frac{1.77}{2.59\times10^{-2}}} = \underline{1.85\times10^{-15} \cong 0}$$

For $T = 1{,}000K$ $\quad kT = 8.63\times10^{-5}\frac{eV}{K}\,1{,}000K = 8.63\times10^{-2}\,eV$

$$N_R = 8.87\times10^{14}\,e^{-1.77/8.63\times10^{-2}} = \underline{1.09\times10^{6}}$$

For $T = 5{,}000K$ $\quad kT = 8.63\times10^{-5}\frac{eV}{K}\,5000K = 0.431\,eV$

$$N_R = 8.87\times10^{14}\,e^{-1.77/0.431} = \underline{1.46\times10^{13}}$$

3. In Problem 2, if the excited atoms that emit visible radiation decay in 10^{-13} s and if only 0.002% of them decay radiatively (quantum yield of 2%), how much power would be radiated from that surface at the aforementioned temperatures? Assume that half of the atoms that radiate emit into the 2π solid angle that would result in their leaving the surface of the material, and assume an average visible photon energy of 2.5 eV. Also compute the total amount of power that could be radiated (over all wavelengths) from the surface at the given temperatures using the Stefan–Boltzmann law (eqn. 6.15). Speculate as to why the two approaches for computing the radiated power are inconsistent at a temperature of 5,000 K.

Using blackbody formula:

$$I_{BB}(\lambda, T, \Delta\lambda) = \frac{3.75\times10^{-25}}{\lambda^5 \left(e^{0.0144/\lambda T} - 1\right)} \Delta\lambda$$

$$\text{Power} = I_{BB}(\lambda, T, \Delta\lambda) \cdot \Delta A$$

$$\Delta A = \text{area} = \pi (0.0005)^2 \, m^2 = 7.85\times10^{-7} \, m^2$$

$$\Delta\lambda = 700 \text{ nm}$$

$$\therefore \quad I_{BB}(\lambda, T, \Delta\lambda) = \frac{3.75\times10^{-25} \, \Delta\lambda}{(500\times10^{-9})^5 \left(e^{0.0144/\lambda T} - 1\right)}$$

$T = 300\,K \qquad \lambda T = 500\times10^{-9} m \; 300K = 1.5\times10^{-4} \, mK$

$T = 1000\,K \qquad \lambda T = 500\times10^{-9} m \; 1000K = 5\times10^{-4} \, mK$

$T = 5000\,K \qquad \lambda T = 500\times10^{-9} m \; 5000K = 2.5\times10^{-3} \, mK$

(continued)

3 (cont.)

$$\text{Power} = I_{BB}(\lambda, T, \Delta\lambda) \cdot \Delta A \qquad (\Delta\lambda \text{ in nm})$$

$$= \frac{1.2\times10^{7}\,\Delta\lambda\,\Delta A}{\left(e^{0.0144/\lambda T}-1\right)} = \frac{6.59\times10^{3}}{\left(e^{0.0144/\lambda T}-1\right)}$$

300K Power $= 1.33\times10^{-38}$ W

1000K Power $= 2.04\times10^{-9}$ W

5000K Power $= 20.8$ W

Look at σT^4 for above (Total power all λ)

300K Power $= 3.6\times10^{-4}$ W

1000K Power $= 4.45\times10^{-2}$ W

5000K Power $= 27.8$ W

If solving for power using $A = 10^{13}/s$

$$\text{Power} = N A h\nu (0.5)(0.00002)$$

For 300K Power $= 7.4\times10^{-26}$ W

For 1000K Power $= 4.36\times10^{-5}$ W

For 5000K Power $= 584$ W ← greater Than σT^4 for <u>all</u> wavelengths

Reason: Collisions reduce population

CH 6

4. How much power is radiated from a 1-mm^2 surface of a body at temperature T when the peak measured wavelength is that of green light at 500 nm?

$$\lambda_m = 500 \times 10^{-9}\ m$$

$$\lambda_m T = 2.898 \times 10^{-3}\ m\text{-}K \qquad (6.16)$$

$$\therefore \quad T = \frac{2.898 \times 10^{-3}\ m\text{-}K}{500 \times 10^{-9}\ m} = 5.796 \times 10^{3}\ K$$

$$= 5{,}796\ K$$

$$I = \epsilon_m \sigma T^4 \ \frac{W}{m^2}$$

Assume $\epsilon_m = 1$ for an ideal black body

$$\text{Then } I = 5.67 \times 10^{-8} \frac{W}{m^2 K^4} (5.796 \times 10^3 K)^4$$

$$= 6.40 \times 10^7 \frac{W}{m^2}$$

for $1\ mm^2 = 10^{-6}\ m^2$

$$\text{Power} = I \times \text{area} = 6.47 \times 10^7 \frac{W}{m^2} 10^{-6} m^2 = \underline{64 W}$$

for $\epsilon_m = 0.5$ Power $= \underline{32 W}$

5. Determine the number of modes in a 1-cm^3 box for frequencies in the visible spectrum between 400 and 700 nm. Compare that value to the number of modes in a sodium streetlamp that emits over a wavelength interval of 3 nm at a center wavelength of 589 nm. Assume that the streetlamp is a cylinder of radius 0.5 cm and length 10 cm.

$$\#\text{ modes up to } \nu_1 = \frac{8\pi \nu_1^3}{3c^3} \times \text{volume}$$

(a) $\therefore$ # modes between 400nm and 700nm is

$$= \left(\frac{8\pi}{3\lambda_{400}^3} - \frac{8\pi}{3\lambda_{700}^3}\right) \times \text{volume}$$

$$= \left[\frac{8\pi}{3(400\times10^{-9})^3} - \frac{8\pi}{3(700\times10^{-9})^3}\right] \times 10^{-6}\,m^3 = 1.11\times10^{14}\text{ modes}$$

(b) Sodium street lamp volume $= (0.1)(0.0025)^2\pi = 1.96\times10^{-6}\,m^3$

$$\left[\frac{8\pi}{3(587.5\times10^{-9})^3} - \frac{8\pi}{3(590.5\times10^{-9})^3}\right] \times 1.96\times10^{-6}\,m^3$$

$= \underline{1.23\times10^{12}\text{ modes in sodium street lamp}}$

Do another way:

$$\frac{d\rho}{d\nu}\Delta\nu \cdot V = \frac{8\pi\nu^2}{c^3}\Delta\nu V = \frac{8\pi c^2}{c^3\lambda^2}\Delta\nu \cdot V$$

$$= \frac{8\pi}{\lambda^4}\Delta\lambda \cdot V$$

$$= \frac{8\pi}{(589\times10^{-9})^4}(3\times10^{-9})(1.96\times10^{-6}) = 1.23\times10^{12}\text{ modes}$$

<u>in street lamp</u>

6. Estimate the number of photons in both the box and the streetlamp of Problem 5 for temperatures of 300 K, 1,000 K, and 5,000 K.

The number of photons is:

$$\frac{(\#\text{ modes}) \times (\text{average energy/mode})}{\text{average energy/photon}}$$

From problem #2

T	kT
300K	2.59×10^{-2} eV
1000K	8.63×10^{-2} eV
5000K	0.431 eV

For box

$$(1.11\times10^{14}\text{ modes})\ \frac{h\nu}{(e^{h\nu/kT}-1)}\ \frac{1}{h\nu}$$

For average photon energy use

$\lambda = 550\text{ nm} \Rightarrow h\nu = 2.26\text{ eV}$

$$\#\text{ photons} = \frac{1.11\times10^{14}}{(e^{h\nu/kT}-1)}$$

T	# photons
300K	1.41×10^{-24}
1,000K	470
5,000K	5.89×10^{11}

For street lamp

Average photon energy $h\nu$ is 2.11 eV for street lamp

$$\#\text{ photons} = \frac{1.23\times10^{12}}{(e^{h\nu/kT}-1)}$$

T	# photons
300K	5.12×10^{-24}
1,000K	29.6
5,000K	9.26×10^{9}

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7. A 100-W incandescent lamp has a tungsten filament composed of a wire, 0.05 cm in diameter and 10 cm in length, that is coiled up to fit within the light bulb. Assume the filament is heated to a temperature of 3,000 K when the light bulb is turned on. How much power (watts) is emitted within the visible spectrum from the filament, assuming that it is emitting as a blackbody? As an approximation, you could divide the visible spectral region into several segments and compute the average contribution from each segment. Then simply add the averages together (instead of trying to integrate the blackbody function over the entire visible spectral range).

$$\text{Power} = I_{BB}(\lambda, T)\,\Delta\lambda \times \text{Area}$$

From (6.44)

$$I_{BB}(\lambda, T) = \frac{3.75\times10^{-25}}{\lambda^5\left(e^{0.0144/\lambda T}-1\right)}$$

λ (nm)
$\Delta\lambda$ (nm)
T (K)

$T = 3{,}000$ K
$\Delta\lambda = 50$ nm
Area $= (0.1\text{ m})\,\pi\,(0.0005)\text{ m} = 1.57\times10^{-4}\text{ m}^2$

λ range	Avg. λ	λT	Power (from above equation)
400-450	425	1.275×10^{-3}	2.64 W
450-500	475	1.425×10^{-3}	4.98 W
500-550	525	1.575×10^{-3}	7.90 W
550-600	575	1.725×10^{-3}	11.10 W
600-650	625	1.875×10^{-3}	14.27 W
650-700	675	2.025×10^{-3}	17.16 W
		Total	58.05 W

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8. Show that Planck's radiation law of (6.39) will lead to the Stefan–Boltzmann relationship of (6.15) if the power radiated over all wavelengths is considered. Determine the coefficients of the Stefan–Boltzmann constant. *Hint:*

$$\int_0^\infty \frac{x^3\,dx}{e^x - 1} = \frac{\pi^4}{15}.$$

Planck's radiation Law: $I_{BB}(\nu) = \frac{2\pi h \nu^3}{c^2} \frac{1}{(e^{h\nu/kT} - 1)}$

Integrate over all frequencies:

$$I_{BB} = \int_0^\infty \frac{2\pi h \nu^3}{c^2} \frac{1}{(e^{h\nu/kT} - 1)} d\nu$$

let $x = h\nu/kT$ $\quad dx = \frac{h}{kT} d\nu \Rightarrow d\nu = \frac{kT}{h} dx$

$\nu = \frac{kTx}{h}$

when $\nu = 0 \rightarrow x = 0$

when $\nu = \infty \rightarrow x = \infty$

$$\therefore \quad I_{BB} = \int_0^\infty \frac{2\pi h}{c^2} \left(\frac{kTx}{h}\right)^3 \frac{1}{(e^x - 1)} \frac{kT}{h} dx$$

$$= \frac{2\pi k^4 T^4}{c^2 h^3} \int_0^\infty \frac{x^3}{e^x - 1} dx = \frac{2\pi k^4 T^4}{c^2 h^3} \cdot \frac{\pi^4}{15}$$

$$= \frac{2\pi^5 k^4}{15 c^2 h^3} T^4 = \underline{\sigma T^4}$$

$$\sigma = \frac{2\pi^5 k^4}{15 c^2 h^3} = \frac{2\pi^5 (1.38 \times 10^{-23}\ J/K)^4}{15 (3 \times 10^8)^2 (6.62 \times 10^{-34}\ J \cdot s)^3}$$

$$= \underline{5.67 \times 10^{-8} \frac{W}{m^2 K^4}}$$

9. An argon ion laser emits 2 W of power at 488.0 nm in a 2-mm–diameter beam. What would be the effective blackbody temperature of the output beam of that laser radiating over the frequency width of the laser transition, given that the laser linewidth is approximately one fifth of the Doppler linewidth? Assume that the laser is operating at an argon gas temperature of 1,500 K and that the laser output is uniform over the width of the beam.

$$u(\nu) = \frac{I(\nu)}{c} = \frac{8\pi h\nu^3}{c^3\left(e^{h\nu/kT}-1\right)}$$

$$I = \frac{2\ \text{W}}{\pi\,(0.001\ \text{m})^2} = 6.37\times10^5\ \text{W/m}^2$$

$$\Delta\nu = \frac{\Delta\nu_D}{5} = \frac{2.7\times10^9}{5} = 5.4\times10^8/\text{s} \quad \left(\begin{array}{l}\text{from}\\ \text{Table 4-1}\end{array}\right)$$

$$\therefore\ I(\nu) = \frac{6.37\times10^5\ \text{W/m}^2}{5.4\times10^8/\text{s}} = 1.18\times10^{-3}\ \frac{\text{W-sec}}{\text{m}}$$

$$e^{h\nu/kT} = 1 + \frac{8\pi h\nu^3}{c^2\, I(\nu)}$$

$$= 1 + \frac{8\pi\ 4.07\times10^{-19}\,\text{J}\,(6.15\times10^{14})^2}{(3\times10^8)^2(1.18\times10^{-3})}$$

$$= 1 + 0.0364 = 1.0364$$

$$\frac{h\nu}{kT} = \ln 1.0364 = 3.575\times10^{-2}$$

$$T = \frac{h\nu}{3.575\times10^{-2}\,k} = \frac{4.07\times10^{-19}\,\text{J}}{3.575\times10^{-2}\ 1.38\times10^{-23}\,\text{J/K}}$$

$$= 8.25\times10^5 = \underline{825{,}000\ \text{K}}$$

10. For the laser in Problem 9, how much power would be required for the stimulated emission rate to equal the radiative decay rate?

For $\frac{B_{ue}\, u(\nu)}{A_{ue}} = 1$ how large must $u(\nu)$ be?

$$u(\nu) = \frac{A_{ue}}{B_{ue}} = \frac{8\pi h \nu^3}{c^3}$$ from (6.52)

$n = 1$

and $\frac{\nu}{c} = \frac{1}{\lambda}$

$$u(\nu) = \frac{8\pi h}{\lambda^3}$$

$$= \frac{8\pi\ 6.6256\times10^{-34}\,\text{J s}}{(488\times10^{-9}\,\text{m})^3} = 1.43\times10^{-13}\,\frac{\text{J s}}{\text{m}^3}$$

$$u(\nu) = \frac{I(\nu)}{c} \Rightarrow I(\nu) = u(\nu)\, c$$

$$I = I(\nu)\,\Delta\nu = u(\nu)\, c\, \Delta\nu$$

$$\Delta\nu = \frac{\Delta\nu_D}{5} = 5.4\times10^8/\text{s}$$ (from Table 4-1)

$$I = 1.43\times10^{-13}\,\frac{\text{J s}}{\text{m}^3}\ 3\times10^8\,\frac{\text{m}}{\text{s}}\ 5.4\times10^8\,\frac{}{\text{s}} = 2.32\times10^4\,\frac{\text{W}}{\text{m}^2}$$

$$\text{Power} = I \times \text{area} = 2.32\times10^4\,\text{W/m}^2\ \pi(10^{-3}\,\text{m})^2$$

$$= 7.29\times10^{-2}\,\text{W} = \underline{72.9\,\text{mW}}$$

compared to 2 W in problem 9

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11. A pulsed and Q-switched Nd:YAG laser is focused to a 200-μm–diameter spot size on a solid flat metal target in an attempt to produce a bright plasma source for microlithography applications. The plasma is observed to radiate uniformly into 2π steradians with a wavelength distribution that is approximately that of a blackbody. The intensity of the Nd:YAG laser is adjusted so that the peak of the blackbody emission occurs at a wavelength of 13.5 nm, the optimum wavelength as a source for EUV microlithography. What is the temperature of the blackbody? What fraction of the total energy radiated by the blackbody would be radiated within the useful bandwidth for microlithography of 0.4 nm centered around the emission maximum? Assume that the plasma doesn't expand significantly during the 10-ns emission time (the duration of the Nd:YAG laser pulses).

(a) $\lambda_m T = 2.898 \times 10^{-3}$ m-K $\qquad \lambda = 13.5$ nm

$$\therefore \quad T = \frac{2.898 \times 10^{-3} \text{ m-K}}{13.5 \times 10^{-9} \text{ m}} = \underline{214{,}667 \text{ K}}$$

(b) $$I = \sigma T^4 = 5.67 \times 10^{-8} \frac{W}{m^2 K^4} (2.147 \times 10^5)^4 K^4$$

$$= 1.20 \times 10^{14} \frac{W}{m^2}$$

$\Delta t = 10^{-8}$ sec

$\Delta A = \pi (10^{-4})^2 = 3.14 \times 10^{-8} m^2$

Total energy $= E_T = I \, \Delta t \, \Delta A$

$$E_T = 1.20 \times 10^{14} \frac{W}{m^2} \, 10^{-8} s \, 3.14 \times 10^{-8} m^2 = \underline{3.77 \times 10^{-2} J}$$

Intensity in 0.4 nm band centered at 13.5 nm is:

$$I_{BB} = \frac{3.75 \times 10^{-25} (0.4 \text{ nm})}{(13.5 \times 10^{-9})^5 \left(e^{0.0144/\lambda T} - 1\right)}$$

$$= 5.86 \times 10^{12} (.4) = 2.34 \times 10^{12} W/m^2$$

Energy in band $= E_{IB} = 2.34 \times 10^{12} W/m^2 \, 10^{-8} s \, 3.14 \times 10^{-8} m^2$

$$= \underline{7.35 \times 10^{-4} J}$$

$$\text{Fraction of Total} = \frac{E_{IB}}{E_T} = \frac{7.35 \times 10^{-4} J}{.0377 J} = 1.95 \times 10^{-2} \simeq \underline{2\%}$$

12. The blackbody spectral distribution curve has a maximum wavelength λ_m that is dependent upon the temperature T of the radiating body. Show that the product $\lambda_m T$ is a constant for any temperature (Wien's law, eqn. 6.16). *Hint:* Use the frequency version of the blackbody radiation formula instead of the wavelength version to show this.

$$I = \frac{2\pi h \nu^3}{c^2}\left(e^{\frac{h\nu}{kT}} - 1\right)^{-1}$$

Let $x = \frac{h\nu}{kT}$ $\quad dx = \frac{h}{kT}\, d\nu$

$$I = 2\pi \frac{h^3 \nu^3}{(kT)^3} \frac{(kT)^3}{h^2}\left(e^{\frac{h\nu}{kT}} - 1\right)^{-1}$$

$$= \frac{2\pi (kT)^3}{h^2} x^3 \left(e^x - 1\right)^{-1}$$

$$\frac{dI}{d\nu} = \frac{dI}{dx}\frac{dx}{d\nu} = \frac{h}{kT}\frac{dI}{dx}$$

but $\frac{dI}{dx} = 3x^2\left(e^x - 1\right)^{-1} - x^3\left(e^x - 1\right)^{2} e^x = 0$

$$3 - x\left(e^x - 1\right)^{-1} e^x = 0$$

$$3 - \frac{x}{(1 - e^{-x})} = 0 \Rightarrow x = 2.82$$

$= \text{constant}$

$$\therefore\ x = \frac{h\nu}{kT} = \text{const} = \frac{hc}{\lambda kT} = \text{const.}$$

$$\therefore\ \underline{\lambda_m T = \text{constant}}$$

13. A blue argon laser beam at 488 nm is propagating around a coliseum as part of a laser light show. The power is measured to be 10 W cw at a specific location with a beam diameter of 5 mm. What is the energy density per unit frequency $u(\nu)$ of the beam at that location?

$$I = \frac{10\ W}{\pi (2.5\times10^{-3})^2\ m^2} = 5.09\times10^5\ \frac{W}{m^2}$$

$$u(\nu) = \frac{I(\nu)}{c} \quad \text{but} \quad I(\nu) = \frac{I}{\Delta\nu}$$

Assume $\Delta\nu =$ Doppler width/5

$$= \frac{2.7\times10^9\ Hz}{5} = 5.4\times10^8\ Hz$$

$$\therefore\ u(\nu) = \frac{I}{\Delta\nu\, c} = \frac{5.09\times10^5\ W/m^2}{5.4\times10^8/s\ \ 3\times10^8\ m/s}$$

$$= \underline{3.14\times10^{-12}\ \frac{J\text{-}s}{m^3}}$$