SOLUTION SET

Chapter 7

CONDITIONS FOR PRODUCING A LASER – POPULATION INVERSIONS, GAIN AND GAIN SATURATION

"LASER FUNDAMENTALS"

Second Edition

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1. Derive the frequency distribution for $A_{ul}(\nu)$, $B_{ul}(\nu)$, and $B_{lu}(\nu)$ for the case of natural broadening. Assume that the total decay rate for level u (when taking into account all frequencies associated with the decay from level u to level l) is A_{ul}, and assume that the total B coefficients are B_{ul} and B_{lu} when integrated over all frequencies associated with the emission of the specific transition from level u to level l.

From (4.64)

$$A_{ul}(\nu) = \frac{\gamma_{ul}^T / 4\pi^2}{(\nu-\nu_0)^2 + \left(\gamma_{ul}^T/4\pi\right)^2} A_{ul}$$

For natural broadening $\gamma_{ul}^T = \sum_i A_{ui} + \sum_j A_{lj}$ (4.35)

$$\therefore A_{ul}(\nu) = \frac{\left[\sum_i A_{ui} + \sum_j A_{lj}\right]/4\pi^2}{(\nu-\nu_0)^2 + \left\{\frac{\left[\sum A_{ui} + \sum A_{lj}\right]}{4\pi}\right\}^2}$$

Also, since $\frac{A_{ul}(\nu)}{B_{ul}(\nu)} = \frac{A_{ul}}{B_{ul}}$ Then

$$B_{ul}(\nu) = \frac{B_{ul}}{A_{ul}} A_{ul}(\nu) = \frac{c^3}{8\pi h \eta^3 \nu^3} A_{ul}(\nu)$$

$$B_{ul}(\nu) = \frac{c^3}{8\pi h \eta^3 \nu^3} \frac{\left[\sum A_{ui} + \sum A_{lj}\right]/4\pi^2}{(\nu-\nu_0)^2 + \left\{\frac{\left[\sum A_{ui} + \sum A_{lj}\right]/4\pi^2}{(\nu-\nu_0)^2 + \left\{\frac{\left[\sum A_{ui} + \sum A_{lj}\right]}{4\pi}\right\}^2}\right\}^2}$$

and $B_{lu}(\nu) = \frac{g_u}{g_l} B_{ul}(\nu)$

2. Derive the frequency distributions of the population densities $N_u(\nu)$ and $N_l(\nu)$, as described by (7.24), for the case of Doppler broadening. Assume that the total population densities in levels u and l are N_u and N_l.

From (7.23)

$$g^D(\nu) = \left[N_u(\nu) - \frac{g_u}{g_l} N_l(\nu) \right] \frac{c^2 A_{ul}}{8\pi \eta^2 \nu^2}$$

but $$N_{u,l}(\nu) = 2\sqrt{\frac{\ln 2}{\pi}} \frac{N_{u,l}}{\Delta\nu_D} \exp\left\{ -\left[\frac{4\ln 2(\nu-\nu_0)^2}{\Delta\nu_D^2} \right] \right\}$$

$$\therefore g^D(\nu) = 2\sqrt{\frac{\ln 2}{\pi}} \frac{c^2 A_{ul}}{8\pi \eta^2 \nu^2} \left[N_u - \frac{g_u}{g_l} N_l \right] \exp\left\{ -\left[\frac{4\ln 2(\nu-\nu_0)^2}{\Delta\nu_D^2} \right] \right\}$$

$$= \sqrt{\frac{\ln 2}{16\pi^3}} \frac{c^2}{\eta^2 \nu^2} A_{ul} \left[N_u - \frac{g_u}{g_l} N_l \right] \exp\left\{ -\left[\frac{4\ln 2(\nu-\nu_0)^2}{\Delta\nu_D^2} \right] \right\}$$

3. Calculate the stimulated emission cross section for the He–Ne laser operating at 632.8 nm, and for the He–Cd laser at 441.6 nm, using the necessary coefficients for those transitions given in earlier chapters. It will be necessary to determine which broadening mechanism is dominant in order to use the appropriate expression for the stimulated emission cross section.

He-Ne laser at 632.8 nm

From Table 4-1 on page 114 it can be seen that the 632.8 nm laser is dominated by Doppler broadening. $\Delta\nu_D = 1.5\times10^9\,Hz$

$$\sigma_{ul}^D(\nu_0) = \sqrt{\frac{\ln 2}{16\pi^3}}\,\frac{\lambda_{ul}^2 A_{ul}}{\Delta\nu_D} = \frac{3.73\times10^{-2}(632.8\times10^{-9})\,3.4\times10^6}{1.5\times10^9}$$

$$= 3.39\times10^{-17}\,m^2$$

He-Cd laser at 441.6 nm

From Table 4-1 $\Delta\nu_N = 4.5\times10^7\,Hz$

$\Delta\nu_D = 1.1\times10^9\,Hz$

Hence Doppler broadening dominates

$$\sigma_{ul}^D(\nu_0) = \frac{3.72\times10^{-2}(441.6\times10^{-9})^2\,1.4\times10^6}{1.1\times10^9}$$

$$= 9.23\times10^{-18}\,m^2$$

4. Calculate the stimulated emission cross section for the Nd:YAG laser using (7.16) together with the necessary values from Table 15-5 for the Nd:YAG laser.

$$\sigma_{ul}^{H}(\nu_0) = \frac{\lambda_{ul}^2 A_{ul}}{4\pi^2 n^2 \Delta\nu_{ul}^{H}}$$

$$\lambda = 1.064\ \mu m$$

$$A_{ul} = 4.3 \times 10^3 / s$$

$$n = 1.82$$

$$\Delta\nu_{ul}^{H} = 1.2 \times 10^{11}$$

$$\therefore \quad \sigma_{ul}^{H}(\nu_0) = \frac{(1.064 \times 10^{-6})^2 4.3 \times 10^3}{4\pi^2 (1.82)^2 (1.2 \times 10^{11})}$$

$$= \underline{3.1 \times 10^{-22}\ m^2}$$

slightly more than a factor of 10 greater than the measured value given in Table 15-5 (page 554)

5. Using the expression derived in this chapter, obtain the saturation intensities of the following lasers.

(a) A helium–neon laser operating at 632.8 nm.

(b) An argon ion laser at 488.0 nm and at 514.5 nm.

(c) A helium–cadmium laser operating at 441.6 nm.

(d) A copper vapor laser at 510.5 nm and 578.2 nm. (F_{sat})

Relevant parameters are available from Chapter 4 or Chapter 14.

From (7.42) $I_{sat} = \dfrac{h\nu_{ul}}{\sigma_{ul}^{H}\,\tau_u}$ $\quad F_{sat} = \dfrac{h\nu_{ul}}{2\,\sigma_{ul}^{H}}$ ← for CVL see (8.92a)

Laser	λ (nm)	$h\nu_{ul}$ (J)	σ_{ul}^{H} (m^2)	τ_u (s)	I_{sat} (W/m^2)
He-Ne	632.8	3.13×10^{-19}	2.46×10^{-15}	3.01×10^{-8}	4.23×10^{3}
Argon+	488.0	4.06×10^{-19}	1.04×10^{-15}	1.0×10^{-8}	3.90×10^{4}
	514.5	3.86×10^{-19}	1.42×10^{-16}	7.58×10^{-8}	3.59×10^{5}
He-Cd	441.6	4.49×10^{-19}	1.54×10^{-16}	1.06×10^{-6}	2.75×10^{3}
					F_{sat} J/m^2
Copper Vapor	510.5	3.89×10^{-19}	5.87×10^{-16}		3.31×10^{-4}
	578.2	3.43×10^{-19}	6.32×10^{-16}		2.71×10^{-4}

$$\sigma_{ul}^{H} = \frac{\lambda_{ul}^{2}\,A_{ul}}{4\pi^{2}\,\Delta\nu_{ul}^{H}} \quad (7.16)$$

Laser	λ(nm)	A_{ul} (s^{-1})	$\Delta\nu_{ul}^{H}$ (s^{-1})	σ_{ul}^{H} (m^2)
He-Ne	632.8	3.4×10^{6}	1.4×10^{7}	2.46×10^{-15}
Ar^+	488.0	7.8×10^{7}	4.54×10^{8}	1.04×10^{-15}
Ar^+	514.5	9.7×10^{6}	4.59×10^{8}	1.42×10^{-16}
He-Cd	441.6	1.4×10^{6}	4.5×10^{7}	1.54×10^{-16}
Copper Vapor	510.5	2.0×10^{6}	2.25×10^{7}	5.87×10^{-16}
Copper Vapor	578.2	1.65×10^{6}	2.21×10^{7}	6.32×10^{-16}

6. What is the ratio of the stimulated emission cross sections at the peak of the emission linewidth for two identical transitions (same A_{ul} and $\Delta\nu_{ul}$), where one is dominated by natural broadening and the other by Doppler broadening?

$$\sigma_{ul}^{N}(\nu_0) = \frac{\lambda^2 A_{ul}}{4\pi^2 \Delta\nu_{ul}^{N}}$$

$$\sigma_{ul}^{D}(\nu_0) = \sqrt{\frac{\ln 2}{16\pi^3}} \frac{\lambda_{ul}^2 A_{ul}}{\Delta\nu_{ul}^{D}}$$

Let $\Delta\nu_{ul}^{N} = \Delta\nu_{ul}^{D} = \Delta\nu$

Let A_{ul} & λ_{ul} be the same for both situations

$$\frac{\sigma_{ul}^{N}}{\sigma_{ul}^{D}} = \frac{\frac{1}{4\pi^2}\frac{\lambda^2 A_{ul}}{\Delta\nu}}{\sqrt{\frac{\ln 2}{16\pi^3}}\frac{\lambda^2 A_{ul}}{\Delta\nu}} = \frac{1}{\sqrt{\pi \ln 2}} = \underline{0.678}$$

7. In Figure 7-6, assume an incremental length of d_a (equal to the diameter of the amplifier) at the left end of the amplifier instead of l_g. Obtain an expression similar to (7.46) in terms of x and L/d_a. Make a plot of L/d versus $\sigma_{ul}^H(\nu)\Delta N_{ul}L$, comparing it to the graph shown in Figure 7-7. You will realize that the incremental length chosen for the end region that initiates the amplification process does not make a significant change in the implications of the graph of Figure 7-7.

$$(N_u \cdot A \cdot d_a)\, A_{ul}\, h\nu_{ul}\, \frac{A}{4\pi^2}\, \frac{e^{\sigma_{ul}^H \Delta N_{ul} L}}{A} = \frac{h\nu_{ul} A_{ul}}{\sigma_{ul}^H}$$

$$N_u \left(\pi \frac{d_a^2}{4}\right) d_a\, \frac{e^{\sigma_{ul}^H N_u L}}{4\pi L^2} = \frac{1}{\sigma_{ul}^H}$$

$$\frac{\sigma_{ul}^H N_u L\, d_a^3}{16 L^3}\, e^{\sigma_{ul}^H N_u L} = 1$$

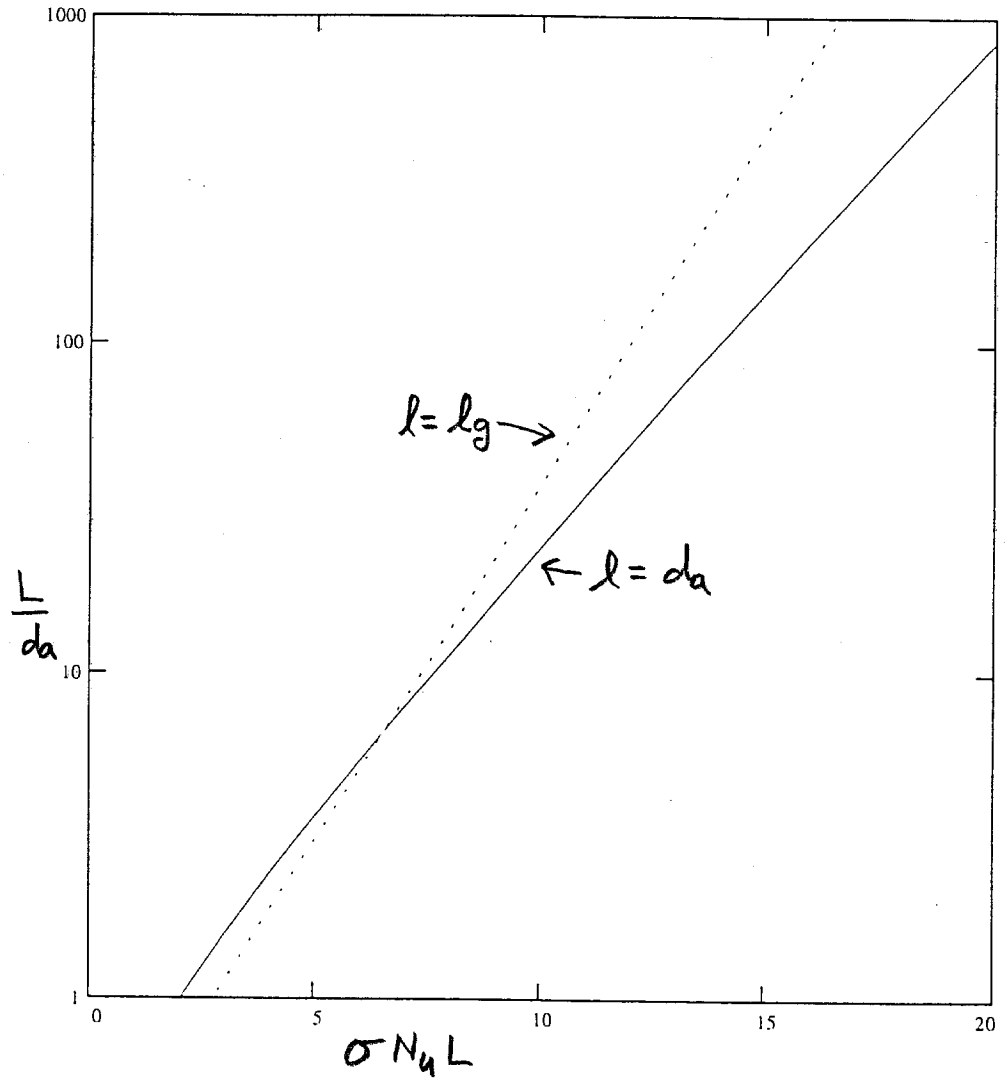
$$\sigma_{ul}^H N_u L\, e^{\sigma_{ul}^H N_u L} = 16 \left(\frac{L}{d_a}\right)^3$$

Plot $\frac{L}{d_a}$ vs $\sigma_{ul}^H N_u L$ for above equation

Compare to plot for $e^{\sigma_{ul}^H N_u L} = 16 \left(\frac{L}{d_a}\right)^2$

(continued)

7 (cont.)



8. For a cavity with mirrors having reflectivities of 98%, what would the minimum gain need to be in order for the laser to reach threshold if the amplifier length is 0.2 m and the mirror separation is 0.4 m? Assume no losses in the cavity other than the mirror transmission losses.

Cavity with $R = 0.98$ or 98%

$L = 0.2\,m$ $\quad d = 0.4\,m$

$$g_{th} = \frac{1}{2L} \ln\left(\frac{1}{R^2}\right)$$

$$= \frac{1}{2(0.2)} \ln \frac{1}{(0.98)^2}$$

$$= \frac{1}{0.4\,m} (4.04 \times 10^{-2})$$

$$= \underline{0.101/m = 10.1\,\%/m}$$

9. **(a)** If you were to design a solid-state laser that reached I_{sat} when pumped, and if the gain medium (a solid-state crystal) were limited to a length of 10 cm, what would the minimum single-pass small-signal exponential gain ($\sigma_{ul}\Delta N_{ul}L$) have to be at line center if the gain duration lasts only 10 ns? Assume a rod diameter of 6 mm and that the mirrors are 100% reflecting and coated onto the ends of the laser rod. Assume also that the crystal has an index of refraction of 2.0.
(b) The linewidth of the homogeneously broadened medium is 100 nm, the laser wavelength at line center is 700 nm, and the radiative transition probability of the laser transition is 10^4 s^{-1}. What population density difference ΔN_{ul} would be required to produce the gain value of part (a)?

Solid State Laser

$L = 10\,cm = 0.1\,m$

$\tau_s = 10\,ns = 10^{-8}\,s$

$d_a = 0.6\,cm = 6\times10^{-3}\,m$

$n = 2.0$

(a) # passes allowed $= \frac{\tau_s c}{L n} = \frac{10^{-8}\,3\times10^{8}}{0.1\,(2.0)} = 15$ passes

$$e^{15 g_0 L} = 16\left(\frac{15L}{d_a}\right)^2$$

$$g_0 = \frac{\ln 16\left(\frac{15L}{d_a}\right)^2}{15L} = \frac{\ln\left[16\left(\frac{1.5}{6\times10^{-3}}\right)^2\right]}{1.5} = 9.21/m$$

$$gL = 9.21/m \times 0.1\,m = \underline{0.921}$$

(b) $\Delta\lambda^H = 100\,nm$ $\quad \lambda_0 = 700\,nm$ $\quad A_{ul} = 10^4/s$

$$\Delta\nu^H = \frac{c\,\Delta\lambda^H}{\lambda_0^2} = \frac{3\times10^8\ 100\times10^{-9}}{(700\times10^{-9})^2} = 6.12\times10^{13}\,Hz$$

$$\sigma_{ul}^H = \frac{\lambda_0^2 A_{ul}}{n^2 4\pi^2 \Delta\nu^H} = \frac{(700\times10^{-9})^2\,10^4}{2^2 4\pi^2\,6.12\times10^{13}} = 5.0\times10^{-25}\,m^2$$

$$g_0 = \sigma_{ul}^H \Delta N_{ul}$$

or $$\Delta N_{ul} = \frac{g_0}{\sigma_{ul}^H} = \frac{9.21/m}{5.0\times10^{-25}\,m^2} = \underline{1.84\times10^{25}/m^3}$$

10. If a copper vapor laser is operated in the double-pass mode (only one mirror), what would the gain and population difference ΔN_{ul} have to be at 510.5 nm in order to reach E_{sat} at the line center in a double pass through the discharge? The gain region of the tube has a length of 1.0 m and a bore size of 25 mm. Assume the single mirror is installed immediately next to the gain medium at one end of the tube. Use the conditions and information of Problem 5(d) if needed. The copper vapor laser operates at a discharge temperature of approximately 1,600°C.

$\lambda = 510.5$ nm $\qquad A_{ul} = 2.0 \times 10^6$

$L = 0.1$ m $\qquad d_a = 25$ mm

$T = 1600°C = 1873 K$

$$\Delta\nu_D = 7.16 \times 10^{-7} \frac{3 \times 10^8}{510.5 \times 10^{-9}} \sqrt{\frac{1873}{63}}$$

$$= 2.3 \times 10^9 \, Hz$$

$$\sigma_{ul}^D = \sqrt{\frac{\ln 2}{16\pi^3}} \frac{\lambda_{ul}^2 A_{ul}}{\Delta\nu_D} = \sqrt{\frac{\ln 2}{16\pi^3}} \frac{(510.5 \times 10^{-9})^2 2.0 \times 10^6}{2.3 \times 10^9}$$

$$= 8.47 \times 10^{-18} \, m^2$$

$$\text{From (7.53)} \left(\frac{1}{\sqrt{\pi \ln 2}}\right) e^{\sigma_{ul}^D N_u L} = 16\left(\frac{L_{sat}}{d_a}\right)^2$$

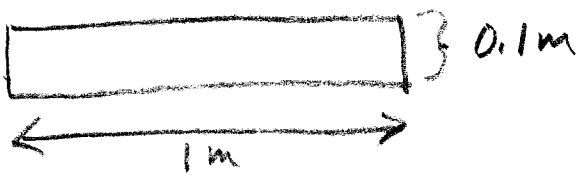
where $N_u \cong \Delta N_{ul}$

$$\Delta N_{ul} = \frac{\ln\left[16\sqrt{\pi \ln 2}\left(\frac{2 L_{sat}}{d_a}\right)^2\right]}{\sigma_{ul}^D (2 L_{sat})} = \frac{\ln\left[16\sqrt{\pi \ln 2}\left(\frac{2(1)}{2.56 \times 10^{-2}}\right)^2\right]}{8.47 \times 10^{-18} \, 2 \, (1)}$$

$$= \underline{7.01 \times 10^{17} / m^3}$$

$$g_0 L = \sigma \Delta N_{ul} L = 8.47 \times 10^{-18} \, 7.01 \times 10^{17} (0.1) = \underline{0.593}$$

11. A high-gain pulsed laser amplifier of length 1 m and diameter 0.01 m has a single 100% reflecting mirror located at one end of the amplifier. The laser is found to have a beam emerging from the other end of the amplifier with an intensity just equal to I_{sat}. If the mirror is removed, by what percentage would the gain have to be increased in order for the beam to reach I_{sat} at the end of the amplifier under the same operating conditions?



$$e^{g2L} = 16\left(\frac{2L}{d}\right)^2 = 16\left(\frac{2(1)}{0.1}\right)^2 = 6.4\times10^3$$

$$g2L = \ln 6.4\times10^3 = \underline{8.76}$$

$$g = \frac{8.76}{2(1)} = 4.38$$

$$e^{gL} = 16\left(\frac{L}{0.1}\right)^2 = 16\left(\frac{1}{0.1}\right)^2 = 1.6\times10^3$$

$$gL = \ln(1.6\times10^3)$$

$$g = \frac{1}{1}\ln(1.6\times10^3) = \underline{7.38}$$

$$\text{gain increase} = \frac{7.38}{4.38} = 1.68 \Rightarrow \underline{68\% \text{ increase}}$$

12. A helium–neon laser with a gain length of 0.2 m is installed in a two-mirror cavity having mirror reflectivities of 99.9% and 98%. What would be the single-pass gain in the laser tube if the discharge current is sufficiently high for the gain to be four times the threshold value? Assume that there are scattering losses of 0.2% per pass within the cavity.

$$L = 0.2\,m$$

$$R_1 = 0.999 \qquad R_2 = 0.98$$

$$a_1 = a_2 = 0.002 \qquad \alpha = 0$$

$$g_0 = 4\,g_{th}$$

$$g_{th} = \frac{1}{2L}\ln\left[\frac{1}{R_1 R_2 (1-a_1)(1-a_2)}\right] + \alpha$$

$$= \frac{1}{0.2}\ln\left[\frac{1}{(0.999)(0.98)(1-0.002)^2}\right]$$

$$= 1.26\times10^{-1}$$

$$g_0 = 4\,g_{th} = 4\,(1.26\times10^{-1}) = 5.04\times10^{-1}$$

$$\frac{I}{I_0} = e^{g_0 L} = e^{5.04\times10^{-1}(.2)} = \underline{1.106}$$

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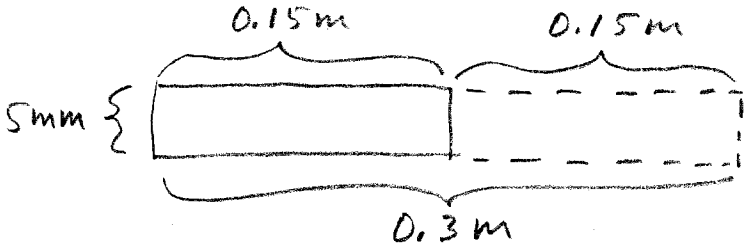
13. A hypothetical laser rod of length 0.15 m and diameter 5 mm is coated with a 100% reflector at one end of the rod and no reflector at the other end (it has an antireflection coating on that end). It operates at a wavelength of 800 nm and is homogeneously broadened. The upper laser level has a lifetime of 200 μs. The pumping flux is such that the laser is at threshold, and the population difference between the upper and lower laser level for that condition is measured to be $1 \times 10^{26}/\text{m}^3$. What is the intensity of the laser at the output end of the rod?



$\lambda = 800\,\text{nm} = 1.55\,\text{eV}$

$\Delta N = 1 \times 10^{26}/\text{m}^3$

$\tau = 2 \times 10^{-4}\,\text{s}$

$$e^{\sigma \Delta N L} = 16\left(\frac{L}{d}\right)^2 = 16\left(\frac{0.30}{0.005}\right)^2 = 5.76 \times 10^4$$

$$\sigma \Delta N L = \ln 5.76 \times 10^4 = 10.96$$

$$\therefore \quad \sigma = \frac{10.96}{\Delta N\, L} = \frac{10.96}{(1 \times 10^{26})(0.3)} = 3.65 \times 10^{-25}\,\text{m}^2$$

$$I_{sat} = \frac{h\nu}{\sigma^H \tau_4} = \frac{1.55}{3.65 \times 10^{-25}\; 2 \times 10^{-4}}$$

$$= 2.12 \times 10^{28}\,\frac{\text{eV}}{\text{m}^2\,\text{sec}}$$

$$= 2.12 \times 10^{28}\,\frac{\text{eV}}{\text{m}^2\,\text{sec}}\;\frac{1.6 \times 10^{-19}\,\text{J}}{\text{eV}} = 3.39 \times 10^9\,\frac{\text{J}}{\text{sec}\,\text{m}^2}$$

$$= 3.39 \times 10^9\,\frac{\text{W}}{\text{m}^2}$$