SOLUTION SET

Chapter 11

LASER CAVITY MODES

"LASER FUNDAMENTALS"

Second Edition

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1. Show that the two expressions for I_t of (11.12) and (11.16) are equivalent; that is, show that

$$I_0 \frac{T^2}{|1 - Re^{i\Phi}|^2} = I_0 \frac{T^2}{(1-R)^2} \frac{1}{1 + F' \sin^2(\Phi/2)}.$$

$$|1 - Re^{i\Phi}|^2 = (1 - Re^{i\Phi})(1 - Re^{-i\Phi})$$

$$= 1 - Re^{i\Phi} - Re^{-i\Phi} + R^2$$

$$= 1 - 2R\cos\Phi + R^2$$

but $\sin^2(\Phi/2) = \dfrac{1 - \cos\Phi}{2}$

or $\cos\Phi = 1 - 2\sin^2(\Phi/2)$

$$\therefore \quad |1 - Re^{i\Phi}|^2 = 1 - 2R\left[1 - 2\sin^2(\Phi/2)\right] + R^2$$

$$= 1 - 2R + 4R\sin^2(\Phi/2) + R^2$$

$$= 1 - 2R + R^2 + 4R\sin^2(\Phi/2)$$

$$= (1-R)^2 + 4R\sin^2\left(\frac{\Phi}{2}\right)$$

$$= (1-R)^2\left[1 + \frac{4R}{(1-R)^2}\sin^2(\Phi/2)\right]$$

Let $F' = \dfrac{4R}{(1-R)^2}$

Then $|1 - Re^{i\Phi}|^2 = (1-R)^2\left[1 + F'\sin^2(\Phi/2)\right]$

$$\therefore \quad I_T = I_0 \frac{T^2}{(1-R)^2} \frac{1}{\left[1 + F'\sin^2(\Phi/2)\right]}$$

2. Compute the FWHM frequency linewidth for a plane-parallel Fabry–Perot interferometer with two identical mirrors, separated by a distance of 1 cm, with air between the mirrors. Assume mirror reflectivities of 99.99%, 95%, and 70%. Determine the cavity Q for these three cases if the output of a GaAs diode laser operating at 800 nm is transmitted through the interferometer.

$d = 0.01\,m \quad \lambda = 800\,nm \quad R = 99.99\%, 95\%, 70\% \quad \eta = 1.0$

$$\Delta\nu = \frac{c}{2\eta d} = \frac{3\times10^8}{2\cdot1\cdot0.01} = 1.5\times10^{10}\,Hz \qquad \nu_0 = \frac{3\times10^8}{800\times10^{-9}} = 3.75\times10^{14}\,Hz$$

$$\Delta\nu_{FWHM} = \frac{\Delta\nu}{F} \qquad F = \frac{\pi\sqrt{R}}{1-R}$$

(a) $R = .9999 \qquad F = 3.141\times10^4$

$$\Delta\nu_{FWHM} = \frac{\Delta\nu}{F} = \frac{1.5\times10^{10}}{3.141\times10^4} = 4.78\times10^5\,Hz$$

$$Q = \frac{\nu_0}{\Delta\nu_{FWHM}} = \frac{3.75\times10^{14}}{4.78\times10^5} = \underline{7.85\times10^8}$$

(b) $R = 0.95 \qquad F = \frac{\pi\sqrt{.95}}{1-.95} = 61.2$

$$\Delta\nu_{FWHM} = \frac{1.5\times10^{10}}{61.2} = 2.45\times10^8\,Hz$$

$$Q = \frac{3.75\times10^{14}}{2.45\times10^8} = \underline{1.53\times10^6}$$

(c) $R = 0.70 \qquad F = \frac{\pi\sqrt{.70}}{1-0.70} = 8.76$

$$\Delta\nu_{FWHM} = \frac{1.5\times10^{10}}{8.76} = 1.71\times10^9\,Hz$$

$$Q = \frac{3.75\times10^{14}}{1.71\times10^9} = \underline{2.19\times10^5}$$

3. A very narrow-frequency 10-mW cw single-mode He–Ne laser beam is transmitted through an air-spaced Fabry–Perot interferometer with the laser wavelength exactly corresponding to one of the 100% transmission peaks of the interferometer, as indicated in Figure 11-5. The reflecting surfaces of the interferometer, separated by a distance of 0.1 m, are dielectric coatings with a reflectivity at the laser wavelength of 99% and no absorption losses. What is the power of the beam inside of the Fabry–Perot cavity? Assume that the laser is instantly shut off in a time of the order of 1 ns or less. How long will it then take for the beam inside the Fabry–Perot cavity to decay to $1/e$ of its steady-state value prior to the shutoff?

(a) $d = 0.1$ m Power $= 10$ mW $R_1 = R_2 = 0.99$

Because the beam is at the peak of the Airy function, the power P_T emerging from the F-P interferometer is the same as the input power or 10 mW. Since the mirror has a transmission of 0.01 and a reflection of 0.99, the power incident (P_I) upon the mirror is $P_I = \frac{P_T}{0.01} = \frac{10\text{ mW}}{0.01} = 1000\text{ mW}$ or 1 W. The power reflected P_R from the mirror is 1000 mW (.99) = 990 mW. Hence the total power inside the cavity is $P_I + P_R = 1000\text{ mW} + 990\text{ mW}$
$= \underline{1990\text{ mW} = 1.99\text{ W}}$

(b) Every time the beam arrives at each mirror, it loses 1% or is reduced to 0.99 of its incident value. After N reflections it will be reduced to a value of $(0.99)^N$. How many passes will it take to be reduced to $1/e$?

$$(0.99)^N = \frac{1}{e} = 0.368 \Rightarrow N = 100$$

Each pass between mirrors takes $\frac{0.1\text{ m}}{3\times10^8\text{ m/s}} = 3.33\times10^{-10}$ s

Hence the decay time is:
$100 \times 3.33\times10^{-10} = \underline{33.3\text{ n sec}}$

4. A 0.1-m–long Nd:YAG laser rod is coated with a 99% reflector on one end and a 95% reflector on the other end. Determine the longitudinal mode spacing. How many modes could be operating if the gain bandwidth is a Lorentzian shape with a FWHM of 0.45 nm and a peak at 1.06 μm? (Assume the laser is mode-locked so that all possible modes are lasing.) Assume that the maximum small-signal gain is 10% at the peak of the emission spectrum and that the cavity losses (in addition to mirror losses) are 2% per pass. The index of refraction of Nd:YAG at 1.06 μm is 1.82. *Note:* Since the transmission losses are different at each mirror, you must consider the total round-trip gain and loss.

$d = 0.1\,m \quad R_1 = 0.99 \quad R_2 = 0.95 \quad n = 1.82$

$$\Delta\nu = \frac{c}{2nd} = \frac{3\times10^8\,m/s}{2(1.82)(0.1)m} = \underline{8.24\times10^8\,Hz}$$

Cavity losses excluding mirror Transmission = 2%/pass

Total round Trip losses $(1-0.99)+(1-0.95)+(0.02)+0.02)$ $= 10\%$

or 5% per pass (averaged)

Hence the gain only exceeds the losses over a band width equivalent to the FWHM of the emission line width.

The FWHM of the emission line is:

$$\Delta\nu = \frac{c}{\lambda^2}\Delta\lambda = \frac{3\times10^8\,m/s\,(0.45\times10^{-9})}{(1.06\times10^{-6})^2} = 1.20\times10^{11}\,Hz$$

Maximum number of longitudinal modes is

$$\left(\frac{1.20\times10^{11}}{8.24\times10^8}+1\right) = \underline{145.6 = 146 \text{ modes}}$$

Note: There is always one more possible mode than there are spaces between modes.

5. A helium–cadmium laser is operating at 441.6 nm with a natural mixture of Cd in the discharge. Assume the laser has a gain of 6% per pass for the isotope 114 at its maximum emission frequency. The laser mirrors are separated by a distance of 0.4 m and the mirrors have reflectivities of 96% and 99.9% at the laser wavelength. Assume that there are scattering losses of 1% per pass through the cavity. Also remember that the index of refraction within a gas laser is 1. How many longitudinal modes would be lasing in this system? Refer to Figure 4-15(a) for information about isotopes of Cd.

$d = 0.4\,m \quad R_1 = 0.96 \quad R_2 = 0.999$

Scattering losses 1% per pass or 0.01/pass

Hence the round trip losses are:

$(1-0.96) + (1-0.999) + 0.01 + 0.01 = 0.061$ or 6.1%

A more accurate way of calculating this is

$[1-(0.96)(0.99)(0.999)(0.99)] = 6.004\times10^{-2} = 6.0\%$

The maximum round trip gain for Cd^{114} is 12%

From Fig. 4-15(a), for Cd^{112} gain $= \frac{24}{28}\,12\% = 10.0\%$

for Cd^{110} gain $= \frac{12.4}{28}\,12\% = 5.2\%$

for Cd^{116} gain $= \frac{7.6}{28}\,12\% = 3.2\%$

Hence only Cd^{112} and Cd^{114} will contribute to lasing. From Fig 4-15(a) the bandwidth for these two isotopes is approximately:

$\Delta\nu_L = 1.46\,GHz + 2\left(\frac{1}{2}\text{ Doppler width}\right)$

$= 1.46 + 1.1 = 2.56\,GHz$

$\Delta\nu_c$ for the laser cavity is

$\Delta\nu_c = \frac{c}{2(.4)} = \frac{3\times10^8}{.8} = 3.75\times10^8\,Hz$

$\#\text{ modes} = \left(\frac{2.56\,GHz}{3.75\times10^8\,Hz} + 1\right) = 7.83 \Rightarrow$ <u>8 modes</u>

6. Show that the solution for a stable mode passing back and forth between two plane-parallel mirrors within a limiting aperture of radius a, as given by (11.51), is the Gaussian function $U(x, y) = e^{-(x^2+y^2)/w^2}$ as expressed in (11.52).

Show that $\gamma\, U(x', y') = C \iint_A U(x,y)\, e^{i k_1 (xx' + yy')}\, dx\, dy$

If $U(x,y)$ is a Gaussian function Then:

Let $U(x,y) = e^{-(x^2+y^2)/w^2}$ let $k_1 = \frac{2}{w^2}$

Then $\gamma\, U(x', y') = C \iint_A e^{-(x^2+y^2)/w^2}\, e^{i \frac{2}{w^2}(xx' + yy')}\, dx\, dy$

change integral: instead of area, go from $-\infty$ to ∞, this still covers the area.

$$\gamma\, U(x', y') = C \int_{-\infty}^{\infty} e^{-x^2/w^2}\, e^{-\frac{2x'}{w^2}x}\, dx \int_{-\infty}^{\infty} e^{-y^2/w^2}\, e^{i\frac{2y'}{w^2}y}\, dy$$

But $\int_{-\infty}^{\infty} e^{-x^2/w^2}\, e^{i k_x x}\, dx \int_{-\infty}^{\infty} e^{-y^2/w^2}\, e^{i k_y y}\, dy$

$$= \pi w^2 e^{-k_x^2 \frac{w^2}{4}}\, e^{-k_y^2 \frac{w^2}{4}}$$

Now let
$k_x = \frac{2}{w^2} x'$
$k_y = \frac{2}{w^2} y'$

Then

$$\gamma\, U(x', y') = C \pi w^2 e^{-\frac{4}{w^4}\frac{w^2}{4} x'^2}\, e^{-\frac{4}{w^4}\frac{w^2}{4} y'^2}$$

$$= C \pi w^2 e^{-(x'^2 + y'^2)/w^2}$$

Q.E.D.

7. Show that, for a TEM_{00} mode passing between two laser mirrors, the transverse displacement – from the center axis, where (according to eqn. 11.53) the intensity reaches a maximum, to the location where the intensity of the mode has a value of $1/e^2$ of the maximum – corresponds to the value w in (11.53) (for a Gaussian mode distribution).

$$E = E_0 e^{-\rho^2/w^2}$$ For TEM_{00} mode of electric field

$$I = I_0 e^{-2\rho^2/w^2}$$

solve for when $I = I_0 e^{-2} = \frac{I_0}{e^2}$

That occurs when $\frac{2\rho^2}{w^2} = 2 \Rightarrow \rho^2 = w^2$

or $\underline{\rho = w}$

8. Write out the expressions for the transverse spatial distribution at the mirrors of the TEM_{11}, TEM_{20}, and TEM_{22} modes.

$$TEM_{11} = \mathcal{U}_{11}(x,y) = H_1\left(\frac{\sqrt{2}x}{w}\right) H_1\left(\frac{\sqrt{2}y}{w}\right) e^{-(x^2+y^2)/w^2}$$

$$= 2\sqrt{2}\,\frac{x}{w}\; 2\sqrt{2}\,\frac{y}{w}\, e^{-(x^2+y^2)/w^2} = \underline{8xy\,e^{-(x^2+y^2)/w^2}}$$

$$TEM_{20} = \mathcal{U}_{20}(x,y) = H_2\left(\frac{\sqrt{2}x}{w}\right) H_0\left(\frac{\sqrt{2}y}{w}\right) e^{-(x^2+y^2)/w^2}$$

$$= 2\left[2\left(\frac{\sqrt{2}x}{w}\right)^2 - 1\right](1)\, e^{-(x^2+y^2)/w^2}$$

$$= 2\left(\frac{4x^2}{w^2} - 1\right) e^{-(x^2+y^2)/w^2} = \underline{\left(\frac{8x^2}{w^2} - 2\right) e^{-(x^2+y^2)/w^2}}$$

$$TEM_{22} = \mathcal{U}_{22}(x,y) = H_2\left(\frac{\sqrt{2}x}{w}\right) H_2\left(\frac{\sqrt{2}y}{w}\right) e^{-(x^2+y^2)/w^2}$$

$$= \underline{\left(\frac{8x^2}{w^2} - 2\right)\left(\frac{8y^2}{w^2} - 2\right) e^{-(x^2+y^2)/w^2}}$$

9. For a confocal cavity with a Fresnel number of 1, compare the losses per pass with those of a plane-parallel resonator for both TEM_{00} and TEM_{01} modes. What are the implications of having a Fresnel number of 1 for a visible laser operating at 500 nm and an infrared laser operating at 10 μm?

confocal cavity with $N = \frac{a^2}{d\lambda} = 1$

From Figure 11-11

	Fractional Power Loss/Pass	
	Plane Parallel	Confocal
TEM_{00}	~ 0.20	~ 0.001
TEM_{01}	~ 0.45	~ 0.01

Plane parallel cavity diffraction loss much greater than for confocal cavity!

10. Assume an argon ion laser is operating at 488.0 nm in a confocal cavity in the TEM_{00} mode with a mirror separation of 0.6 m. Assume that the gain medium is 0.4 m long. What must the gain coefficient be for the laser to operate at threshold with mirror reflectivities of 99.99% at the laser wavelength? Assume that the beam radius (the effective limiting aperture) for the laser beam is 0.5 mm at each mirror. Assume also that the only losses in the cavity are the diffraction losses at the mirrors. (The mirrors are mounted at the ends of the laser discharge tube such that there are no Brewster angle windows.)

$d = 0.6\,m \qquad \lambda = 488.0\,nm \qquad a = 0.0005\,m$

$$N = \frac{a^2}{\lambda d} = \frac{(0.0005)^2}{488\times10^{-9}(0.6)} = 0.85$$

From Figure 11-11
For confocal cavity TEM_{00} mode
The diffraction loss/pass is $\sim 0.0025 = a_1, a_2$

From (7.60)

$$g_{th} = \frac{1}{2L}\ln\left[\frac{}{R_1 R_2 (1-a_1)(1-a_2)}\right] + \alpha$$

but $\alpha \cong 0 \quad L = 0.4$ and $R_1 = R_2 = 0.9999$

$$\therefore \quad g_{th} = \frac{1}{2(0.4)}\ln\left[\frac{1}{(0.9999)^2(1-0.0025)^2}\right]$$

$$= 6.5\times10^{-3} = \underline{0.0065/m}$$

11. A CO_2 laser is operating at 10.6 μm in a confocal resonator having a mirror separation of 0.5 m such that both the TEM_{00} and TEM_{11} modes are lasing. What is the distance between the maximum intensity points of diagonally opposite lobes of the TEM_{11} mode at the mirrors?

$$I_{11} = |u_{11}|^2 = \left[\frac{8xy}{w^2} e^{-(x^2+y^2)/w^2}\right]^2$$

$$= \frac{64x^2y^2}{w^4} e^{-2(x^2+y^2)/w^2}$$

For diagonal directions, $y = x$ or

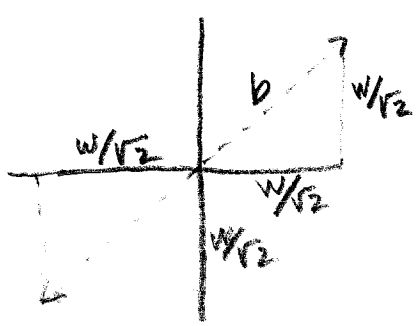
$$I = 64 \frac{x^4}{w^4} e^{-4x^2/w^2}$$

For maximum intensity, Take derivative w.r.t. x:

$$\frac{dI}{dx} = \frac{64}{w^4}\left(4x^3 e^{-4x^2/w^2} - \frac{8x^5}{w^2} e^{-4x^2/w^2}\right) = 0$$

$$= \frac{64}{w^4} 4x^3 e^{-4x^2/w^2}\left(1 - \frac{2x^2}{w^2}\right) = 0$$

$$x^2 = \frac{w^2}{2} \quad x = \frac{w}{\sqrt{2}}$$



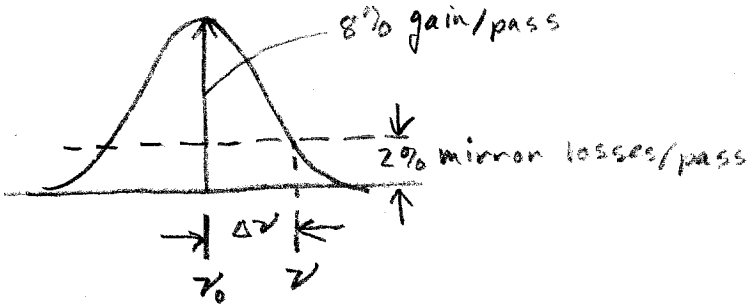
$$b = \sqrt{\left(\frac{w}{\sqrt{2}}\right)^2 + \left(\frac{w}{\sqrt{2}}\right)^2} = \sqrt{\frac{2w^2}{2}} = w$$

but for confocal resonator at mirrors $w = \sqrt{\frac{\lambda d}{\pi}}$

$$\therefore \text{distance} = 2b = 2w = 2\sqrt{\frac{\lambda d}{\pi}} = 2\sqrt{\frac{10.6\times10^{-6}(0.5)}{\pi}}$$

$$= \underline{2.60\times10^{-3}\ \text{m}}$$

12. A Nd:YAG laser has a homogeneously broadened gain profile with a FWHM of 120 GHz. The laser is flashlamp pumped to operate cw with a single-pass small-signal gain at the center of the gain profile of 8%. Mirrors are coated on the ends of the 0.1-m–long Nd:YAG rod with a reflectivity of 98% each. What is the maximum number of longitudinal modes that could be lasing? (Assume that the laser is operating under mode-locked conditions such that all possible modes are lasing.)



Losses are 2%/pass

Therefore the maximum net gain is 6%/pass at the emission line center

The gain profile is homogeneous and can therefore be expressed as:

$$g = C_1 \frac{\gamma/4\pi^2}{(\nu-\nu_0) + (\gamma/4\pi)^2}$$

when $\nu = \nu_0$,

$$g = C_1 \frac{4}{\gamma} = 0.08$$

or $C_1 = \frac{.08}{4}\gamma = .02\gamma$

Solve for $\Delta\nu = \nu - \nu_0$ when $g = 0.02$

$$0.02 = .02\gamma \frac{\gamma/4\pi^2}{\Delta\nu^2 + (\gamma/4\pi)^2}$$

or $$\frac{\gamma^2/4\pi^2}{\Delta\nu^2 + (\gamma/4\pi)^2} = 1 \Rightarrow \Delta\nu^2 + \frac{\gamma^2}{16\pi^2} = \frac{\gamma^2}{4\pi^2}$$

or $$\Delta\nu^2 = \frac{3}{16}\frac{\gamma^2}{\pi^2}$$

but $\gamma = 2\pi\Delta\nu^H = 2\pi\, 1.2\times10^{11}\,Hz = 7.54\times10^{11}\,Hz$

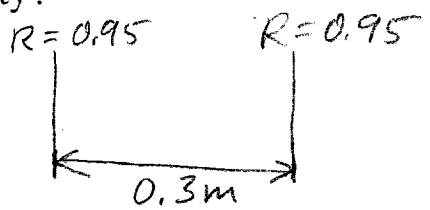
$$\therefore \Delta\nu = \sqrt{\frac{3}{16}}\frac{\gamma}{\pi} = \sqrt{\frac{3}{16}}\frac{7.54\times10^{11}}{\pi} = 1.04\times10^{11}\,Hz$$

$\therefore$ Max gain width is $2\Delta\nu = 2.08\times10^{11}\,Hz$

Mode spacing is $\Delta\nu_M = \frac{c}{2nd} = \frac{3\times10^8}{2(1.82)(0.1)} = 8.24\times10^8\,Hz$

$$\#\text{ modes} = \left(\frac{2.08\times10^{11}}{8.24\times10^8} + 1\right) = \underline{253.4 = 254 \text{ modes}}$$

13. An argon ion laser beam operating at 488 nm is passed through a Fabry–Perot cavity in air at normal incidence with no reduction in its intensity. The cavity has a mirror separation of 0.3 m, mirror reflectivities of 95% at the laser wavelength, and no absorption losses. What is the minimum distance the mirror separation would have to be changed in order to reduce the transmitted intensity to one half of the input intensity?



(11.33) $$\Delta\nu_{FWHM} = \frac{c(1-R)}{2\pi n d\sqrt{R}} = \frac{3\times10^8(1-0.95)}{2\pi(1)(0.3)\sqrt{.95}} = 8.16\times10^6\,Hz$$

$$\nu = \frac{nc}{2nd} \qquad \Delta\nu = \frac{nc}{2n}\left(-\frac{1}{d^2}\right)\Delta d$$

$$|\Delta d| = \frac{2nd^2\Delta\nu}{nc}$$

$$\nu_1 = \frac{c}{\lambda_1} = \frac{n_1 c}{2nd_1} \Rightarrow n_1 = \frac{2nd_1 c}{\lambda_1 c} = \frac{2nd_1}{\lambda_1}$$

$$\therefore \Delta d_1 = \frac{2nd_1^2\Delta\nu_1}{n_1 c} = \frac{2nd_1^2\Delta\nu_1}{2nd_1/\lambda\, c} = \frac{d_1\Delta\nu_1\lambda_1}{c}$$

$$d_1 = 0.3m \quad \Delta\nu_1 = \Delta\nu_{FWHM}/2 = 4.08\times10^6\,Hz$$

$$\lambda_1 = 488\times10^{-9}\,m$$

$$\Delta d_1 = \frac{(0.3)(4.08\times10^6\,Hz)(488\times10^{-9}\,m)}{3\times10^8\,m/s} = 1.99\times10^{-9}\,m$$

$$\underline{\cong 2\,nm}$$